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ELEMENTS OF PLANE AND SOLID
GEOMETRY,

AND OF

PLANE AND SPHERICAL
TRIGONOMETRY.

TO WHICH IS ADDED

MENSURATION, SURVEYING, AND NAVIGATION.

BY

GERARDUS BEEKMAN DOCHARTY, LL.D.,

PROFESSOR OF MATHEMATICS IN THE COLLEGE OF THE CITY OF NEW YORK; AUTHOR OF A "PRACTICAL AND COMMERCIAL ARITHMETIC," THE "INSTITUTES OF ALGEBRA," "ELEMENTS OF ANALYTICAL GEOMETRY AND OF THE DIFFERENTIAL AND INTEGRAL CALCULUS."

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of New York.

TO

HORACE WEBSTER, LL.D.,

PRESIDENT OF THE COLLEGE OF THE CITY OF NEW YORK,

AS A TRIBUTE OF RESPECT FOR THE FAITHFUL PERFORMANCE OF THE DUTIES BELONGING TO HIS OFFICE,

A TOKEN OF SINCERE REGARD FOR HIS DEVOTION TO THE CAUSE OF SOUND EDUCATION, AND

A MEMORIAL OF PERSONAL FRIENDSHIP CEMENTED BY A DAILY INTERCOURSE OF MANY YEARS,

This Volume

IS MOST SINCERELY DEDICATED BY

THE AUTHOR.

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PREFACE.

It has been my endeavor, in this volume, to furnish the student with every aid which can be afforded consistently with securing the end which mathematical studies are designed to attain. I have not attempted to lessen the amount of thought and attention which are necessary for him to put forth, but only to prevent him from wasting that thought and attention.

The most perfect demonstration of a mathematical theorem is that which, while rigorously exact, is expressed in the fewest and simplest words. The "Elements of Euclid," with some exceptions, are acknowledged to be absolutely perfect in this respect. Taking these for a basis and model, I have reduced the essential features of Geometry to so small a compass that I have been enabled to bring that science, as well as its cognate and dependent ones, Trigonometry, Mensuration, Surveying, and Navigation, within the compass of a single volume.

In the chapters on Trigonometry, Mensuration, Surveying, and Navigation, the student will find much that will, by exercising his analytical powers, fit him for the successful study of the higher branches of Mathematics.

I have prepared this treatise with a special view to meet the wants of students in the College of the City of New York, who are early trained to understand and appreciate the subtle methods of demonstration occasionally used in it.

The great success which has followed the introduction of my Algebra, Analytics, and Differential and Integral Calculus, is some guarantee for the value of this volume; and the high standing which the students of this free college have taken among engineers, architects, and educators of our country, furnishes also a living commentary on the method of instruction which is employed therein.

GERARDUS BEEKMAN DOCHARTY.

College of the City of New York, July, 1866.

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GEOMETRY.

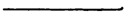
BOOK I.

DEFINITIONS.

GEOMETRY is a science which treats of the properties and relations of quantities or magnitudes having extension.

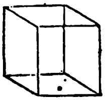
1. A Point is that which has position, but no magnitude nor dimensions; neither length, breadth, nor thickness.

2. A Line is length without breadth or thickness.



3. A Surface or Superficies is an extension or a figure of two dimensions, length and breadth; but without thickness.

4. A Body or Solid is a figure of three dimensions, namely, length, breadth, and depth, or thickness.



5. Lines are either Right or Curved, or Mixed of these two.

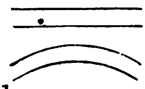
6. A Right Line or Straight Line lies all in the same direction, between its extremities, and is the shortest distance between two points. From one point to another only one straight line can be drawn.

When a Line is mentioned simply, it means a Right Line.

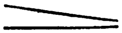
7. A Curve continually changes its direction between its extreme points.

8. Lines are either Parallel, Oblique, Perpendicular, or Tangential.

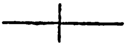
9. Parallel Lines are always at the same perpendicular distance; and they never meet, though ever so far produced.



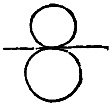
10. Oblique Lines change their distance, and would meet if produced on the side of the least distance.



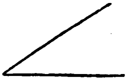
11. One line is Perpendicular to another when it inclines not more on the one side than the other, or when the angles on both sides of it are equal.



12. A Line or Circle is Tangential, or is a Tangent to a Circle or other curve, when it touches it without cutting, although both are produced.

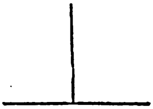


13. An Angle is the inclination or opening of two lines having different directions and meeting in a point.



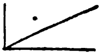
14. Angles are Right or Oblique, Acute or Obtuse.

15. A Right Angle is that which is made by one line perpendicular to another. Or, when the angles on each side are equal to one another, they are right angles.

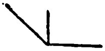


16. An Oblique Angle is that which is made by two oblique lines, and is either less or greater than a right angle.

17. An Acute Angle is less than a right angle.



18. An Obtuse Angle is greater than a right angle.



19. Superficies are either Plane or Curved.

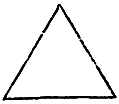
20. A Plane Superficies, or a Plane, is that with which a right line may every way coincide. Or, if the line touch the plane in two points, it will touch it in every point. But if not, it is curved.

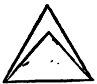
21. Plane figures are bounded either by right lines or curves.

22. Plane figures that are bounded by right lines have names according to the number of their sides or of their angles; for they have as many sides as angles; the least number being three.

23. A figure of three sides and angles is called a Triangle; and it receives particular denominations from the relations of its sides and angles.

24. An Equilateral Triangle is that whose three sides are all equal.





25. An Isosceles Triangle is that which has two sides equal.

26. A Scalene Triangle is that whose three sides are all unequal.

27. A Right-angled Triangle is that which has one right angle.

28. Other Triangles are Oblique-angled, and are either obtuse or acute.

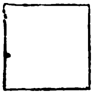
29. An Obtuse-angled Triangle has one obtuse angle.

30. An Acute-angled Triangle has all its three angles acute.

31. A figure of Four sides and angles is called a Quadrangle, or a Quadrilateral.

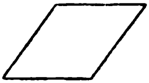
32. A Parallelogram is a quadrilateral which has both its pairs of opposite sides parallel. And it takes the following particular names, viz., Rectangle, Square, Rhombus, Rhomboid.

33. A Rectangle is a parallelogram having a right angle.



34. A Square is an equilateral rectangle, having its length and breadth equal, or all its sides equal, and all its angles equal.

35. A Rhomboid is an oblique-angled parallelogram.



36. A Rhombus is an equilateral rhomboid, having all its sides equal, but its angles oblique:

37. A Trapezium is a quadrilateral which has not its opposite sides parallel.

38. A Trapezoid has only one pair of opposite sides parallel.

39. A Diagonal is a line joining any two opposite angles of a quadrilateral.

40. Plane figures that have more than four sides are, in general, called Polygons; and they receive other particular names, according to the number of their sides or angles. Thus,

41. A Pentagon is a polygon of five sides; a Hexagon, of six sides; a Heptagon, seven; an Octagon, eight; a Non-

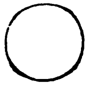
agon, nine; a Decagon, ten; an Undecagon, eleven; and a Dodecagon, twelve sides.

42. A Regular Polygon has all its sides and all its angles equal. If they are not both equal, the polygon is Irregular.

43. An Equilateral Triangle is also a Regular Figure of three sides, and the Square is one of four: the former being also called a Trigon, and the latter a Tetragon.

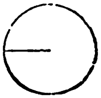
44. Any figure is equilateral when all its sides are equal: and it is equiangular when all its angles are equal. When both these are equal, it is a regular figure.

45. A Circle is a plane figure bounded by a curve line, called the Circumference, which is every where equidistant from a certain point within, called its Centre.

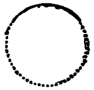


The circumference itself is often called a circle, and also the Periphery.

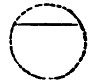
46. The Radius of a circle is a line drawn from the centre to the circumference.



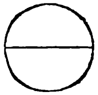
47. An Arc of a circle is any part of the circumference.



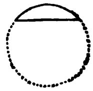
48. A Chord is a right line joining the extremities of an arc.



49. The Diameter of a circle is the chord which passes through the centre.

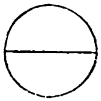


50. A Segment is any part of a circle bounded by an arc and its chord.

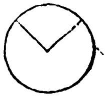


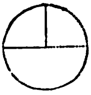
51. A Semicircle is half the circle, or a segment cut off by a diameter.

The half circumference is sometimes called the Semicircle.

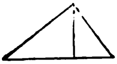


52. A Sector is any part of a circle which is bounded by an arc and two radii drawn to its extremities.





53. A Quadrant, or Quarter of a circle, is a sector having a quarter of the circumference for its arc, and its two radii are perpendicular to each other. A quarter of the circumference is sometimes called a Quadrant.



54. The Height or Altitude of a figure is a perpendicular let fall from an angle, or its vertex, to the opposite side, called the base.

55. In a Right-angled Triangle, the side opposite the right angle is called the Hypothenuse; and the other two sides are called the Legs, and sometimes the Base and Perpendicular.

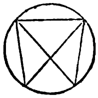
56. When an Angle is denoted by three letters, of which one stands at the angular point, and the other two on the two sides, that which stands at the angular point is read in the middle.

57. The Circumference of every circle is supposed to be divided into 360 equal parts, called Degrees; and each degree into 60 Minutes, each minute into 60 Seconds, and so on. Hence a semicircle contains 180 degrees, and a quadrant 90 degrees.

58. The Measure of an angle is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is estimated by the number of degrees contained in that arc.

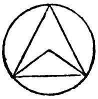
59. Lines or Chords are said to be equidistant from the centre of a circle when perpendiculars drawn to them from the centre are equal.

60. And the right line on which the Greater Perpendicular falls is said to be farther from the centre.

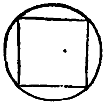


61. An Angle in a Segment is that which is contained by two lines drawn from any point in the arc of the Segment to the two extremities of that arc.

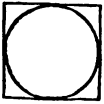
62. An Angle on a segment or an arc is that which is contained by two lines drawn from any point in the opposite or supplementary part of the circumference to the extremities of the arc, and containing the arc between them.



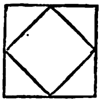
63. An Angle at the circumference is that whose angular point or summit is any where in the circumference. And an angle at the centre is that whose angular point is at the centre.



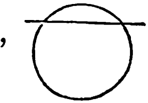
64. A Right-lined figure is Inscribed in a circle, or the circle Circumscribes it, when all the angular points of the figure are in the circumference of the circle.



65. A Right-lined figure Circumscribes a circle, or the circle is Inscribed in it, when all the sides of the figure touch the circumference of the circle.



66. One Right-lined figure is Inscribed in another, or the latter Circumscribes the former, when all the angular points of the former are placed in the sides of the latter.



67. A Secant is a line that cuts a circle, lying partly within, and partly without it.

68. Two Triangles, or other right-lined figures, are said to be mutually equilateral when all the sides of the one are equal to the corresponding sides of the other, each to each; and they are said to be mutually equiangular when the angles of the one are respectively equal to those of the other.

69. Identical figures are such as are both mutually equilateral and equiangular, or that have all the sides and all the angles of the one respectively equal to all the sides and all the angles of the other, each to each; so that if the one figure were applied to, or laid upon the other, all the sides of the one would exactly fall upon and cover all the sides of the other, the two becoming, as it were, but one and the same figure.

70. Similar figures are those that have all the angles of the one equal to all the angles of the other, each to each, and the sides about the equal angles proportional.

71. The Perimeter of a figure is the sum of all its sides taken together.

72. A Proposition is something which is either proposed to be done or to be demonstrated, and is either a problem or a theorem.

73. A Problem is something proposed to be done.

74. A Theorem is something proposed which requires a demonstration.

75. A Lemma is something which is premised or demonstrated, in order to render what follows more easy.

76. A Corollary is a consequent truth, gained immediately from some preceding truth or demonstration.

77. A Scholium is a remark or observation made upon something going before it.

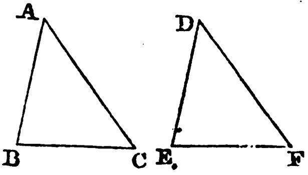
AXIOMS.

1. Things which are equal to the same thing are equal to each other.
2. When equals are added to equals, the wholes are equal.
3. When equals are taken from equals, the remainders are equal.
4. When equals are added to unequals, the wholes are unequal.
5. When equals are taken from unequals, the remainders are unequal.
6. Things which are double of the same thing, or equal things, are equal to each other.
7. Things which are halves of the same thing are equal.
8. Every whole is equal to all its parts taken together, or it is greater than any of its parts.
9. Things which coincide, or fill the same space, are identical, or mutually equal in all their parts.
10. All right angles are equal to one another.
11. Angles that have equal measures, or arcs, are equal.

THEOREM I.

If two triangles have two sides and the included angle in the one equal to two sides and the included angle in the other, the triangles will be identical, or equal in all respects.

In the two triangles ABC, DEF, if the side AC be equal to the side DF, and the side BC equal to the side EF, and the angle C equal to the angle F, then will the two triangles be identical, or equal in all respects.



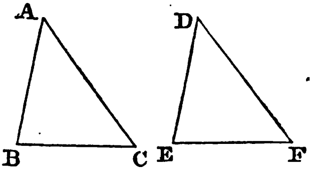
For, conceive the triangle ABC to be applied to, or placed on, the triangle DEF, in such a manner that the point C may coincide with the point F, and the side AC with the side DF, which is equal to it.

Then, since the angle F is equal to the angle C (by hyp.), the side BC will fall on the side EF. Also, because AC is equal to DF, and BC equal to EF (by hyp.), the point A will coincide with the point D, and the point B with the point E; consequently the side AB will coincide with the side DE. Therefore the two triangles are identical, and have all their other corresponding parts equal (ax. 9), namely, the side AB equal to the side DE, the angle A to the angle D, and the angle B to the angle E. Q. E. D.

THEOREM II.

When two triangles have two angles and the included side in the one equal to two angles and the included side in the other, the triangles are identical, or have their other sides and angles equal.

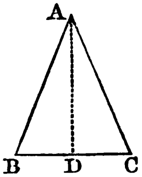
Let the two triangles ABC, DEF, have the angle A equal to the angle D, the angle B equal to the angle E, and the side AB equal to the side DE; then these two triangles will be identical.



For, conceive the triangle ABC to be placed on the triangle DEF in such manner that the side AB may fall exactly on the equal side DE. Then, since the angle A is equal to the angle D (by hyp.), the side AC must fall on the side DF; and, in like manner, because the angle B is equal to the angle E, the side BC must fall on the side EF. Thus the three sides of the triangle ABC will be exactly placed on the three sides of the triangle DEF; consequently the two triangles are identical (ax. 9), having the other two sides AC, BC, equal to the two DF, EF, and the remaining angle C equal to the remaining angle F. Q. E. D.

THEOREM III.

In an isosceles triangle, the angles at the base are equal. Or, if a triangle have two sides equal, their opposite angles will also be equal.



If the triangle ABC have the side AC equal to the side AB, then will the angle B be equal to the angle C.

For, conceive the angle A to be bisected, or divided into two equal parts, by the line AD, making the angle BAD equal to the angle CAD.

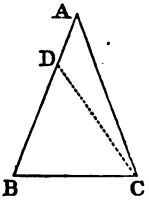
Then the two triangles ACD, ABD, have two sides and the contained angle of the one, equal to two sides and the contained angle of the other, *viz.*, the side AC equal to AB, the angle BAD equal to CAD, and the side AD common; therefore these two triangles are identical, or equal in all respects (th. 1); and, consequently, the angle C equal to the angle B. Q. E. D.

Corol. 1. Hence the line which bisects the vertical angle of an isosceles triangle bisects the base, and is also perpendicular to it.

Corol. 2. Hence, too, it appears that every equilateral triangle is also equiangular, or has all its angles equal.

THEOREM IV.

When a triangle has two of its angles equal, the sides opposite to them are also equal.



Let ABC be a triangle, having the angle B equal to the angle C; then will the side AC be equal to the side AB.

For, if they are not equal, suppose AB > AC, and take BD = AC, and join DC.

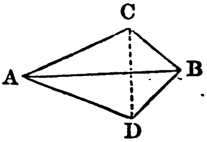
Then, in the two triangles BCD, ABC, we have the side BD = AC, by supposition; the side BC common, and the included angle CBD of the one triangle equal to the included angle BCA of the other triangle. Therefore the triangle BDC is equal to the triangle ABC (th. 1). That is, a part is equal to the whole, which is impossible (ax. 8).

Hence there is no inequality of the sides AB and AC; that is, they are equal, and the triangle is isosceles. Q. E. D.

Corollary. Hence every equiangular triangle is also equilateral.

THEOREM V.

When two triangles have all the three sides in the one equal to all the three sides in the other, the triangles are identical, or have also their three angles equal, each to each.



Let the two triangles ABC, ABD, have their three sides respectively equal, viz., the side AB equal to AB, AC to AD, and BC to BD; then shall the two triangles be identical, or have their angles equal, *viz.*, those angles that are opposite to the equal sides; that is, the angle BAC to the angle BAD, the angle ABC to the angle ABD, and the angle C to the angle D.

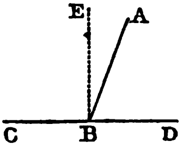
For, conceive the two triangles to be joined together by their longest equal sides, and draw the line CD.

Then, in the triangle ACD, because the side AC is equal to AD (by hyp.), the angle ACD is equal to the angle ADC (th. 3). In like manner, in the triangle BCD, the angle BCD is equal to the angle BDC, because the side BC is equal to BD. Hence, then, the angle ACD being equal to the angle ADC, and the angle BCD to the angle BDC, by equal additions the sum of the two angles ACD, BCD, is equal to the sum of the two ADC, BDC (ax. 2); that is, the whole angle ACB equal to the whole angle ADB.

Since, then, the two sides AC, CB, are equal to the two sides AD, DB, each to each (by hyp.), and their contained angles ACB, ADB, also equal, the two triangles ABC, ABD, are identical (th. 1), and have the other angles equal, *viz.*, the angle BAC to the angle BAD, and the angle ABC to the angle ABD. Q. E. D.

THEOREM VI.

When one line meets another, the angles which it makes on the same side of the other are together equal to two right angles.



Let the line AB meet the line CD; then will the two angles ABC, ABD, taken together, be equal to two right angles.

For, first, when the two angles ABC, ABD, are equal to each other, they are both of them right angles (def. 15).

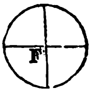
But when the angles are unequal, suppose BE drawn perpendicular to CD. Then, since the two angles EBC, EBD, are right angles (def. 15), and the angle EBD is equal to the two angles EBA, ABD, together (ax. 8), the three angles EBC, EBA, and ABD are equal to two right angles.

But the two angles EBC, EBA, are together equal to the angle ABC (ax. 8). Consequently, the two angles ABC, ABD, are also equal to two right angles. Q. E. D.

Corol. 1. Hence also, conversely, if the two angles ABC, ABD, on both sides of the line AB, make up together two right angles, then CB and BD form one continued right line CD.

Corol. 2. Hence all the angles which can be made, at any point B, by any number of lines, on the same side of the right line CD, are, when taken all together, equal to two right angles.

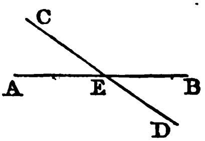
Corol. 3. And as all the angles that can be made on the other side of the line CD are also equal to two right angles, therefore all the angles that can be made quite round a point B, by any number of lines, are equal to four right angles.



Corol. 4. Hence, also, the whole circumference of a circle, being the sum of the measures of all the angles that can be made about the centre F (def. 58), is the measure of four right angles. Consequently, a semicircle, or 180 degrees, is the measure of two right angles; and a quadrant, or 90 degrees, the measure of one right angle.

THEOREM VII.

When two lines intersect each other, the opposite angles are equal.



Let the two lines AB, CD, intersect in the point E; then will the angle AEC be equal to the angle BED, and the angle AED be equal to the angle CEB.

For, since the line CE meets the line AB, the two angles AEC, BEC, taken together, are equal to two right angles (th. 6).

In like manner, the line BE, meeting the line CD, makes the two angles BEC, BED, equal to two right angles.

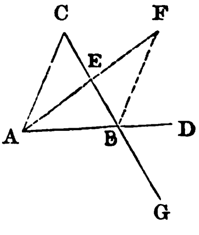
Therefore, the sum of the two angles AEC, BEC, is equal to the sum of the two BEC, BED (ax. 1).

And if the angle BEC, which is common, be taken away from both these, the remaining angle AEC will be equal to the remaining angle BED (ax. 3).

And, in like manner, it may be shown that the angle AED is equal to the opposite angle BEC.

THEOREM VIII.

When one side of a triangle is produced, the exterior angle is greater than either of the two interior opposite angles.



Let ABC be a triangle, having the side AB produced to D; then will the outward angle CBD be greater than either of the inward opposite angles A or C.

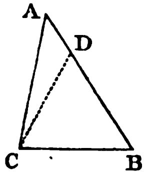
For, conceive the side BC to be bisected in the point E, and draw the line AE, producing it till EF be equal to AE; and join BF.

Then, since the two triangles AEC, BEF, have the side AE = the side EF, and the side CE = the side BE (by suppos.), and the included or opposite angles at E also equal (th. 7), therefore those two triangles are equal in all respects (th. 1), and have the angle C = the corresponding angle EBF. But the angle CBD is greater than the angle EBF; consequently, the said outward angle CBD is also greater than the angle C.

In like manner, if CB be produced to G, and AB be bisected, it may be shown that the outward angle ABG, or its equal CBD, is greater than the other angle A.

THEOREM IX.

The greater side of every triangle is opposite to the greater angle, and the greater angle opposite to the greater side.



Let ABC be a triangle, having the side AB greater than the side BC; then will the angle ACB, opposite the greater side AB, be greater than the angle A, opposite the less side CB.

For, on the greater side AB, take the part BD equal to the less side BC, and join CD. Then, since ACD is a triangle, the exterior angle BDC is greater than the interior opposite angle A (th. 8). But the angle BCD is equal to the said exterior angle BDC, because BD is equal to BC (th. 3). Consequently, the angle BCD also is greater than the angle A. And since the angle BCD is only a part of ACB, much more must the whole angle ACB be greater than the angle A. Q. E. D.

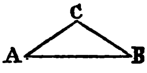
Again, conversely, if the angle C be greater than the angle A, then will the side AB, opposite the former, be greater than the side BC, opposite the latter.

For, if AB be not greater than BC, it must be either equal to it, or less than it. But it can not be equal, for then the angle C would be equal to the angle A (th. 3), which it is not by the supposition. Neither can it be less, for then the angle C would be less than the angle A, by the former part of this, which is also contrary to the supposition. The side AB, then, being neither equal to AC nor less than it, must necessarily be greater. Q. E. D.

THEOREM X.

The sum of any two sides of a triangle is greater than the third side.

Let ABC be a triangle: then will the sum of two of its sides, as AB+BC, be greater than the third side AC.



For the straight line AC is the shortest distance between the points A and C (def. 6); hence AB+BC>AC.

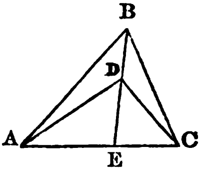
Corol. If from both members of the inequality we subtract the side BC, we shall have

$$AB>AC-BC:$$

that is, *the difference of any two sides of a triangle is less than the third side.*

THEOREM XI.

If from a point within a triangle two lines be drawn to the extremities of either side, their sum will be less than the sum of the other two sides of the triangle.



Let ABC be a triangle, and D any point within it. Draw CD, DB, to the extremities of the side BC; then will BD+CD be less than AB+AC.

Produce BD to E; then BA+AE>BE. To both members of this inequality add EC; then will BA+AE+EC>BE+EC; or, which is the same thing, BA+AC>BE+EC.

Again, in the triangle DEC, we have

$$DE+EC>DC.$$

To both members of this inequality add BD; then DE+BD+EC>DC+BD, or

$$BE+EC>BD+DC.$$

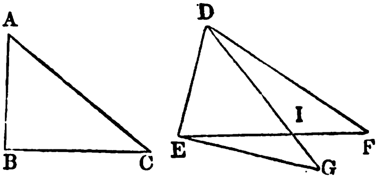
Hence, much more is

$$BA+AC>BD+DC, \text{ or}$$
$$BD+DC<BA+AC.$$

THEOREM XII.

If two triangles have two sides of the one equal to two sides of the other, but the included angles unequal, the third sides will be unequal, and the greater will be in that triangle which has the greater included angle.

Let ABC, DEF be two triangles in which AB=DE, AC



$= DF$, $BAC < EDF$. Then will $EF > BC$.

For at the point D make the angle EDG equal to the angle BAC; take DG equal to AC, and join GE. Then will the triangle DEG equal the triangle ABC (th. 1), and $EG = BC$.

But $EG < EI + IG$, and $DF < DI + IF$ (Prop. 10).

By addition of these inequalities, $EG + DF < EI + IF + DI + IG$; or, $EG + DF < EF + DG$.

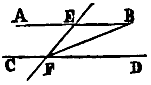
Taking away the equals DF and DG from the members of the last inequality, there remains $EG < EF$. But $EG = BC \therefore BC < EF$. Q. E. D.

If the point G fall within instead of without the triangle DEF, we should have $DG + GE < DF + EF$ (th. 11); and, taking away the equals DG and DF, there remains $EG < EF$. If the point G fall on EF, the theorem is evident.

The converse of this proposition is also true, viz., that if two sides of one triangle be equal to two sides of another, and the third sides unequal, the angle opposite the smaller third side will be less than the one opposite the larger. For if the angle were greater, by the above proposition the third side must be greater; and if it were equal, it must (by th. 1) be equal. But the third side is neither greater nor equal; therefore the angle opposite, being neither greater than nor equal to the angle of the other triangle, must be less.

THEOREM XIII.

When a line intersects two parallel lines, it makes the alternate angles equal to each other.



Let the line EF cut the two parallel lines AB, CD; then will the angle AEF be equal to the alternate angle EFD.

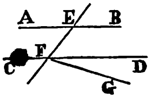
For if they are not equal, one of them must be greater than the other; let it be EFD, for instance, which is the greater, if possible; and conceive the line FB to be drawn, cutting off the part or angle EFB equal to the angle AEF, and meeting the line AB in the point B.

Then, since the exterior angle AEF, of the triangle BEF, is greater than the interior opposite angle EFB (th. 8); and since these two angles also are equal (by the constr.), it follows that those angles are both equal and unequal at the same time, which is impossible. Therefore the angle EFD is not unequal to the alternate angle AEF, that is, they are equal to each other. Q. E. D.

Corol. Right lines which are perpendicular to one of two parallel lines, are also perpendicular to the other.

THEOREM XIV.

When a line, cutting two other lines, makes the alternate angles equal to each other, those two lines are parallel.



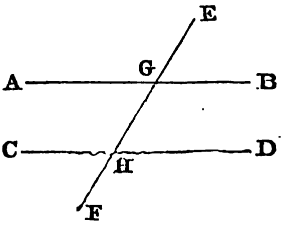
Let the line EF, cutting the two lines AB, CD, make the alternate angles AEF, DFE, equal to each other; then will AB be parallel to CD.

For, if they be not parallel, let some other line, as FG, be parallel to AB. Then, because of these parallels, the angle AEF is equal to the alternate angle EFG (th. 13). But the angle AEF is equal to the angle EFD (by hyp.). Therefore the angle EFD is equal to the angle EFG (ax. 1); that is, a part is equal to the whole, which is impossible. Therefore no line but CD can be parallel to AB. Q. E. D.

Corol. Those lines which are perpendicular to the same line, are parallel to each other.

THEOREM XV.

When a line cuts two parallel lines, the exterior angle is equal to the interior opposite one on the same side; and the two interior angles on the same side are together equal to two right angles.



Let the line EF cut the two parallel lines AB, CD; then will the exterior angle EGB be equal to the interior opposite angle GHD, on the same side of the line EF; and the two interior angles BGH, GHD, taken together, will be equal to two right angles.

For, since the two lines AB, CD, are parallel, the angle AGH is equal to the alternate angle GHD (th. 13). But the angle AGH is equal to the opposite angle EGB (th. 7). Therefore the angle EGB is also equal to the angle GHD (ax. 1). Q. E. D.

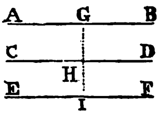
Again, because the two adjacent angles EGB, BGH, are together equal to two right angles (th. 6); of which the angle EGB has been shown to be equal to the angle GHD; therefore the two angles BGH, GHD, taken together, are also equal to two right angles.

Corol. 1. And, conversely, if one line meeting two other lines make the angles on the same side of it equal, those two lines are parallels.

Corol. 2. If a line cutting two other lines make the sum of the two inward angles on the same side less than two right angles, those two lines will not be parallel, but will meet each other when produced.

THEOREM XVI.

Those lines which are parallel to the same line are parallel to each other.

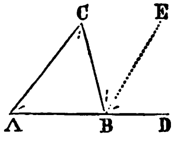


Let the lines AB, CD, be each of them parallel to the line EF; then shall the lines AB, CD, be parallel to each other.

For, let the line GI be perpendicular to EF. Then will this line be also perpendicular to both the lines AB, CD (corol., th. 13), and, consequently, the two lines AB, CD, are parallels (corol., th. 14). Q. E. D.

THEOREM XVII.

When one side of a triangle is produced, the exterior angle is equal to both the interior opposite angles taken together.

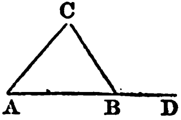


Let the side AB, of the triangle ABC, be produced to D; then will the exterior angle CBD be equal to the sum of the two interior opposite angles A and C.

For, conceive BE to be drawn parallel to the side AC of the triangle. Then BC, meeting the two parallels AC, BE, makes the alternate angles C and CBE equal (th. 13). And AD, cutting the same two parallels AC, BE, makes the interior and exterior angles on the same side, A and EBD, equal to each other (th. 14). Therefore, by equal additions, the sum of the two angles A and C is equal to the sum of the two CBE and EBD; that is, to the whole angle CBD (by ax. 2). Q. E. D.

THEOREM XVIII.

In any triangle, the sum of all the three angles is equal to two right angles.



Let ABC be any plane triangle; then the sum of the three angles A+B+C is equal to two right angles.

For, let the side AB be produced to D. Then the exterior angle CBD is equal to the sum of the two interior opposite angles A+C (th. 17). To each of these equals add the interior angle B; then will the sum of the three interior angles A+B+C be equal to the sum of the two adjacent angles ABC+CBD (ax. 2). But the sum of these two last adjacent angles is equal to two right angles (th. 6). Therefore, also, the sum of the three angles of the triangle A+B+C is equal to two right angles (ax. 1). Q. E. D.

Corol. 1. If two angles in one triangle be equal to two angles in another triangle, the third angles will also be equal (ax. 3), and the two triangles equiangular.

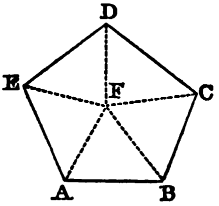
Corol. 2. If one angle in one triangle be equal to one angle in another, the sums of the remaining angles will also be equal (ax. 3).

Corol. 3. If one angle of a triangle be right, the sum of the other two will also be equal to a right angle, and each of them singly will be acute, or less than a right angle.

Corol. 4. The two least angles of every triangle are acute, or each less than a right angle.

THEOREM XIX.

In any figure whatever, the sum of all the interior angles, taken together, is equal to twice as many right angles, wanting four, as the figure has sides.

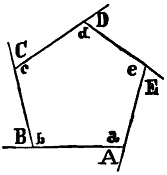


Let ABCDE be any figure; then the sum of all its interior angles, A+B+C+D+E, is equal to twice as many right angles, wanting four, as the figure has sides.

For, from any point F, within it, draw lines, FA, FB, FC, &c., to all the angles, dividing the polygon into as many triangles as it has sides. Now the sum of the three angles of each of these triangles is equal to two right angles (th. 18); therefore the sum of the angles of all the triangles is equal to twice as many right angles as the figure has sides. But the sum of all the angles about the point F, which are so many of the angles of the triangles, but no part of the interior angles of the polygon, is equal to four right angles (corol. 3, th. 6), and must be deducted out of the former sum. Hence it follows that the sum of all the interior angles of the polygon alone, A+B+C+D+E, is equal to twice as many right angles as the figure has sides, wanting the said four right angles. Q. E. D.

THEOREM XX.

When every side of any figure is produced out, the sum of all the exterior angles thereby made is equal to four right angles.

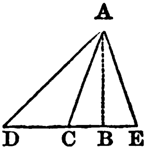


Let A, B, C, &c., be the exterior angles of any polygon, made by producing all the sides; then will the sum A+B+C+D+E, of all those exterior angles, be equal to four right angles.

For every one of these exterior angles, together with its adjacent interior angle, make up two right angles, as A+a equal to two right angles, being the two angles made by one line meeting another (th. 6). And there being as many exterior or interior angles as the figure has sides, therefore the sum of all the interior and exterior angles is equal to twice as many right angles as the figure has sides. But the sum of all the interior angles, with four right angles, is equal to twice as many right angles as the figure has sides (th. 19); therefore the sum of all the interior and all the exterior angles is equal to the sum of all the interior angles and four right angles (by ax. 1). From each of these take away all the interior angles, and there remain all the exterior angles equal to four right angles (by ax. 3). Q. E. D.

THEOREM XXI.

A perpendicular is the shortest line that can be drawn from a given point to an indefinite line. And, of any other lines drawn from the same point, those that are nearest the perpendicular are less than those more remote.



If AB, AC, AD, &c., be lines drawn from the given point A to the indefinite line DE, of which AB is perpendicular, then shall the perpendicular AB be less than AC, and AC less than AD, &c.

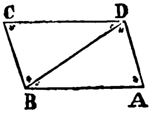
For, the angle B being a right one, the angle C is acute (by cor. 3, th. 18), and therefore less than the angle B. But the less angle of a triangle is subtended by the less side (th. 9). Therefore the side AB is less than the side AC.

Again, the angle ACB being acute, as before, the adjacent angle ACD will be obtuse (by th. 6); consequently, the angle D is acute (corol. 3, th. 18), and therefore is less than the angle C. And since the less side is opposite to the less angle, therefore the side AC is less than the side AD. Q. E. D.

Corol. A perpendicular is the least distance of a given point from a line.

THEOREM XXII.

The opposite sides and opposite angles of any parallelogram are equal to each other; and the diagonal divides it into two equal triangles.



Let ABCD be a parallelogram, of which the diagonal is BD; then will its opposite sides and opposite angles be equal to each other, and the diagonal BD will divide it into two equal parts, or triangles.

For, since the sides AB and DC are parallel, as also the sides AD and BC (defin. 32), and the line BD meets them, therefore the alternate angles are equal (th. 13), namely, the angle ABD to the angle CDB, and the angle ADB to the angle CBD. Hence the two triangles, having two angles in the one equal to two angles in the other, have also their third angles equal (cor. 1, th. 18), namely, the angle A equal to the angle C, which are two of the opposite angles of the parallelogram.

Also, if to the equal angles ABD, CDB, be added the equal angles CBD, ADB, the wholes will be equal (ax. 2), namely, the whole angle ABC to the whole ADC, which are the other two opposite angles of the parallelogram. Q. E. D.

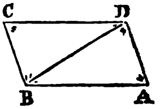
Again, since the two triangles are mutually equiangular, and have a side in each equal, viz., the common side BD, therefore the two triangles are identical (th. 2), or equal in all respects, namely, the side AB equal to the opposite side DC, and AD equal to the opposite side BC, and the whole triangle ABD equal to the whole triangle BCD. Q. E. D.

Corol. 1. Hence, if one angle of a parallelogram be a right angle, all the other three will also be right angles, and the parallelogram a rectangle.

Corol. 2. Hence, also, the sum of any two adjacent angles of a parallelogram is equal to two right angles.

THEOREM XXIII.

Every quadrilateral whose opposite sides are equal is a parallelogram, or has the opposite sides parallel.



Let ABCD be a quadrangle, having the opposite sides equal, namely, the side AB equal to DC, and AD equal to BC; then shall these equal sides be also parallel, and the figure a parallelogram.

For, let the diagonal BD be drawn. Then, the triangles ABD, CBD, being mutually equilateral (by hyp.), they are also mutually equiangular (th. 5), or have their corresponding angles equal; consequently, the opposite sides are parallel (th. 14); viz., the side AB parallel to DC, and AD parallel to BC, and the figure is a parallelogram. Q. E. D.

THEOREM XXIV.

Those lines which join the corresponding extremes of two equal and parallel lines are themselves equal and parallel.

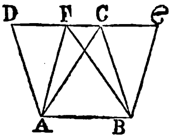
Let AB, DC, be two equal and parallel lines; then will the lines AD, BC, which join their extremes, be also equal and parallel. [See the fig. above.]

For, draw the diagonal BD. Then, because AB and DC are parallel (by hyp.), the angle ABD is equal to the alternate angle BDC (th. 13). Hence, then, the two triangles, having two sides and the contained angles equal, viz., the side AB equal to the side DC, and the side BD common, and the contained angle ABD equal to the contained angle BDC, they have the remaining sides and angles also respectively equal (th. 1); consequently, AD is equal to BC, and also parallel to it (th. 14). Q. E. D.

THEOREM XXV.

Parallelograms, as also triangles, standing on the same base and between the same parallels, are equal to each other.

Let ABCD, ABEF, be two parallelograms, and ABC, ABF, two triangles, standing on the same base, AB, and between the same parallels AB, DE; then will the parallelo-



gram ABCD be equal to the parallelogram ABEF, and the triangle ABC equal to the triangle ABF.

For, since the line DE cuts the two parallels AF, BE, and the two AD, BC, it makes the angle E equal to the angle AFD, and the angle D equal to the angle BCE (th. 15); the two triangles ADF, BCE, are therefore equiangular (cor. 1, th. 18); and having the two corresponding sides AD, BC, equal (th. 22), being opposite sides of a parallelogram, these two triangles are identical, or equal in all respects (th. 2). If each of these equal triangles, then, be taken from the whole space ABED, there will remain the parallelogram ABEF in the one case, equal to the parallelogram ABCD in the other (by ax. 3).

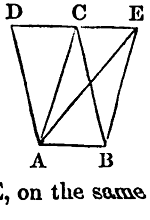
Also, the triangles ABC, ABF, on the same base AB, and between the same parallels, are equal, being the halves of the said equal parallelograms (th. 22). Q. E. D.

Corol. 1. Parallelograms or triangles having the same base and altitude are equal. For the altitude is the same as the perpendicular or distance between the two parallels, which is every where equal, by the definition of parallels.

Corol. 2. Parallelograms or triangles having equal bases and altitudes are equal. For, if the one figure be applied with its base on the other, the bases will coincide or be the same, because they are equal; and so the two figures, having the same base and altitude, are equal.

THEOREM XXVI.

If a parallelogram and a triangle stand on the same base and between the same parallels, the parallelogram will be double the triangle, or the triangle half the parallelogram.



Let ABCD be a parallelogram, and ABE a triangle, on the same base AB, and between the same parallels AB, DE; then will the parallelogram ABCD be double the triangle ABE, or the triangle half the parallelogram.

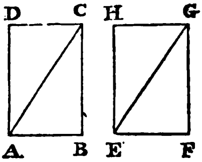
For, draw the diagonal AC of the parallelogram, dividing it into two equal parts (th. 22). Then, because the triangles ABC, ABE, on the same base and between the same parallels are equal (th. 25), and because the one triangle ABC is half the parallelogram ABCD (th. 22), the other equal triangle ABE is also equal to half the same parallelogram ABCD. Q. E. D.

Corol. 1. A triangle is equal to half a parallelogram of the same base and altitude, because the altitude is the perpendicular distance between the parallels, which is every where equal, by the definition of parallels.

Corol. 2. If the base of a parallelogram be half that of a triangle of the same altitude, or the base of the triangle be double that of the parallelogram, the two figures will be equal to each other.

THEOREM XXVII.

Rectangles that are contained by equal lines are equal to each other.



Let BD, FH, be two rectangles, having the sides AB, BC, equal to the sides EF, FG, each to each; then will the rectangle BD be equal to the rectangle FH.

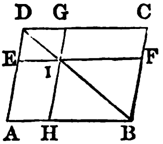
For, draw the two diagonals AC, EG, dividing the two parallelograms each into two equal parts. Then the two triangles ABC, EFG, are equal to each other (th. 1), because they have the two sides AB, BC, and the contained angle B, equal to the two sides EF, FG, and the contained angle F (by hyp.). But these equal triangles are the halves of the respective rectangles. And because the halves or the triangles are equal, the wholes, or the rectangles DB, HF, are also equal (by ax. 6). Q. E. D.

Corol. The squares on equal lines are also equal; for every square is a species of rectangle.

THEOREM XXVIII.

The complements of the parallelograms which are about the diagonal of any parallelogram are equal to each other.

Let AC be a parallelogram, BD a diagonal, EIF parallel to AB or DC, and GIH parallel to AD or BC, making AI,

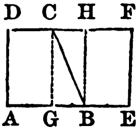


IC, complements to the parallelograms EG, HF, which are about the diagonal DB; then will the complement AI be equal to the complement IC.

For, since the diagonal DB bisects the three parallelograms AC, EG, HF (th. 22), therefore, the whole triangle DAB being equal to the whole triangle DCB, and the parts DEI, IHB, respectively equal to the parts DGI, IFB, the remaining parts AI, IC, must also be equal (by ax. 3). Q. E. D.

THEOREM XXIX.

A trapezoid or quadrilateral having two sides parallel is equal to half a parallelogram whose base is the sum of these two sides, and its altitude the perpendicular distance between them.



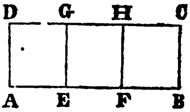
Let ABCD be the trapezoid, having its two sides AB, DC, parallel; and in AB produced take BE equal to DC, so that AE may be the sum of the two parallel sides; produce DC also, and let EF, GC, BH, be all three parallel to AD. Then is AF a parallelogram of the same altitude with the trapezoid ABCD, having its base AE equal to the sum of the parallel sides of the trapezoid; and it is to be proved that the trapezoid ABCD is equal to half the parallelogram AF.

Now, since triangles or parallelograms of equal bases and altitude are equal (corol. 2, th. 25), the parallelogram DG is equal to the parallelogram HE, and the triangle CGB equal to the triangle CHB; consequently, the line BC bisects, or equally divides, the parallelogram AF, and ABCD is the half of it. Q. E. D.

THEOREM XXX.

The sum of all the rectangles contained under one whole line, and the several parts of another line any way divided, is equal to the rectangle contained under the two whole lines.

Let AD be the one line, and AB the other, divided into the parts AE, EF, FB; then will the rectangle contained by AD and AB be equal to the sum of the rectangles of



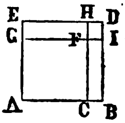
AD and AE, and AD and EF, and AD and FB; thus expressed, AD . AB= AD . AE+AD . EF+AD . FB.

For, make the rectangle AC of the two whole lines AD, AB, and draw EG, FH, perpendicular to AB, or parallel to AD, to which they are equal (th. 22). Then the whole rectangle AC is made up of all the other rectangles AG, EH, FC. But these rectangles are contained by AD and AE, EG and EF, FH and FB; which are equal to the rectangles of AD and AE, AD and EF, AD and FB, because AD is equal to each of the two, EG, FH. Therefore the rectangle AD . AB is equal to the sum of all the other rectangles AD . AE, AD . EF, AD . FB. Q. E. D.

Corol. If a right line be divided into any two parts, the square on the whole line is equal to both the rectangles of the whole line and each of the parts.

THEOREM XXXI.

The square of the sum of two lines is greater than the sum of their squares by twice the rectangle of the said lines. Or, the square of a whole line is equal to the sum of the squares of its two parts, together with twice the rectangle of those parts.



Let the line AB be the sum of any two lines AC, CB; then will the square of AB be equal to the squares of AC, CB, together with twice the rectangle of AC . CB. That is, $AB^2 = AC^2 + CB^2 + 2AC \cdot CB$.

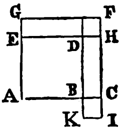
For, let ABDE be the square on the sum or whole line AB, and ACFG the square on the part AC. Produce CF and GF to the other sides at H and I.

From the lines CH, GI, which are equal, being each equal to the sides of the square AB or BD (th. 22), take the parts CF, GF, which are also equal, being the sides of the square AF, and there remains FH equal to FI, which are also equal to DH, DI, being the opposite sides of the parallelogram. Hence the figure HI is equilateral; and it has all its angles right ones (corol. 1, th. 22); it is therefore a square on the line FI, or the square of its equal CB. Also the figures EF, FB, are equal to two rectangles under AC and CB, because GF is equal to AC, and FH or FI equal to CB. But the whole square AD is made up of the four figures, viz., the two squares AF, FD, and the two equal rectangles EF, FB. That is, the square of AB is equal to the squares of AC, CB, together with twice the rectangle of AC, CB. Q. E. D.

Corol. Hence, if a line be divided into two equal parts, the square of the whole line will be equal to four times the square of half the line.

THEOREM XXXII.

The square of the difference of two lines is less than the sum of their squares by twice the rectangle of the said lines.



Let AC, BC, be any two lines, and AB their difference; then will the square of AB be less than the squares of AC, BC, by twice the rectangle of AC and BC. Or, $AB^2 = AC^2 + BC^2 - 2AC \cdot BC$.

For, let ABDE be the square on the difference AB, and ACFG the square on the line AC. Produce ED to H; also produce DB and HC, and draw KI, making BI the square of the other line BC.

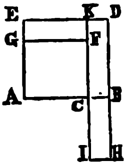
Now it is visible that the square AD is less than the two squares AF, BI, by the two rectangles EF, DI. But GF is equal to the one line AC, and GE or FH is equal to the other line BC; consequently the rectangle EF, contained under EG and GF, is equal to the rectangle of AC and BC.

Again: FH being equal to CI or BC or DH, by adding the common part HC, the whole HI will be equal to the whole FC, or equal to AC; and, consequently, the figure DI is equal to the rectangle contained by AC and BC.

Hence the two figures EF, DI, are two rectangles of the two lines AC, BC; and, consequently, the square of AB is less than the squares of AC, BC, by twice the rectangle AC . BC. Q. E. D.

THEOREM XXXIII.

The rectangle under the sum and difference of two lines is equal to the difference of the squares of those lines.[*]



Let AB, AC, be any two unequal lines; then will the difference of the squares of AB, AC, be equal to a rectangle under their sum and difference. That is,

$$AB^2-AC^2=\overline{AB+AC}\,.\,\overline{AB-AC}.$$

For, let ABDE be the square of AB, and ACFG the square of AC. Produce DB till BH be equal to AC; draw HI parallel to AB or ED, and produce FC both ways to I and K.

Then the difference of the two squares AD, AF, is evidently the two rectangles EF, KB. But the rectangles EF, BI, are equal, being contained under equal lines; for EK and BH are each equal to AC, and GE is equal to CB, being each equal to the difference between AB and AC, or their equals AE and AG. Therefore the two EF, KB, are equal to the two KB, BI, or to the whole KH; and, consequently, KH is equal to the difference of the squares AD, AF. But KH is a rectangle contained by DH, or the sum of AB and AC, and by KD, or the difference of AB and AC; therefore the difference of the squares of AB, AC, is equal to the rectangle under their sum and difference. Q. E. D.

THEOREM XXXIV.

In any right-angled triangle, the square of the hypothenuse is equal to the sum of the squares of the other two sides.

Let ABC be a right-angled triangle, having the right angle A; then will the square of the hypothenuse CB be equal to the sum of the squares of the other two sides AC, AB. Or $CB^2=AC^2+AB^2$.

For, on CB describe the square CE, and on AC, AB, the

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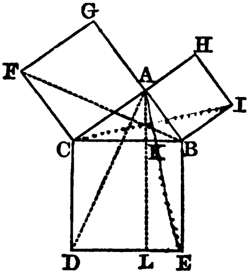
[*] This and the two preceding theorems are evinced algebraically by the three expressions,

$$(a+b)^2=a^2+2ab+b^2=a^2+b^2+2ab\,;$$
$$(a-b)^2=a^2-2ab+b^2=a^2+b^2-2ab\,;$$
$$(a+b)\,(a-b)=a^2-b^2.$$

squares CG, BH; then draw AK parallel to CD or BE; and join CI, BF, AD, AE.



Now, because the line AC meets the two AG, AB, so as to make two right angles, these two form one straight line GB (corol. 1, th. 6). And because the angle FCA is equal to the angle DCB, being each a right angle, or the angle of a square; to each of these equals add the common angle BCA, so will the whole angle or sum FCB be equal to the whole angle or sum ACD. But the line FC is equal to the line AC, and the line CB to the line CD, being sides of the same squares; so that the two sides FC, CB, and their included angle FCB, are equal to the two sides CA, CD, and the contained angle ACD, each to each; therefore the whole triangle CFB is equal to the whole triangle ACD (th. 1).

But the square CG is double the triangle CFB, on the same base FC, and between the same parallels FC, GB (th. 26); in like manner, the parallelogram CK is double the triangle ACD, on the same base CD, and between the same parallels CD, AK. And since the doubles of equal things are equal (by ax. 6), therefore the square CG is equal to the parallelogram CL.

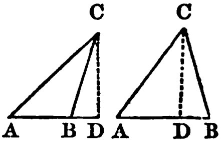
In like manner, the other square BH is proved equal to the other parallelogram BL; consequently, the two squares CG and BH together are equal to the two parallelograms CL and BL together, or to the whole square CE. That is, the sum of the two squares on the two less sides is equal to the square on the greatest side. Q. E. D.

Corol. 1. Hence the square of either of the two less sides is equal to the difference of the squares of the hypothenuse and the other side (ax. 3), or equal to the rectangle contained by the sum and difference of the said hypothenuse and other side (th. 33).

Corol. 2. Hence, also, if two right-angled triangles have two sides of the one equal to two corresponding sides of the other, their third sides will also be equal, and the triangles identical.

THEOREM XXXV.

In any triangle, the difference of the squares of the two sides is equal to the difference of the squares of the segments of the base, or of the two lines or distances included between the extremes of the base and the perpendicular.



Let ABC be any triangle, having CD perpendicular to AB; then will the difference of the squares of AC, BC, be equal to the difference of the squares of AD, BD; that is, $AC^2-BC^2=AD^2-BD^2$.

$$\left.\begin{array}{l}\text{For, since } AC^2 \text{ is equal to } AD^2+CD^2\\ \text{and} \qquad\quad BC^2 \text{ is equal to } BD^2+CD^2\end{array}\right\}\text{(by th. 34),}$$

therefore the difference between AC^2 and BC^2 is equal to the difference between AD^2+CD^2 and BD^2+CD^2, or equal to the difference between AD^2 and BD^2, by taking away the common square CD^2. Q. E. D.

Corol. The rectangle of the sum and difference of the two sides of any triangle is equal to the rectangle of the sum and difference of the distances between the perpendicular and the two extremes of the base, or equal to the rectangle of the base and the difference or sum of the segments, according as the perpendicular falls within or without the triangle. That is,

$$(AC+BC).(AC-BC)=(AD+BD).(AD-BD).$$

Or $(AC+BC).(AC-BC)=AB\,(AD-BD)$ in the 2d fig., and $(AC+BC).(AC-BC)=AB.(AD+BD)$ in the 1st fig.

THEOREM XXXVI.

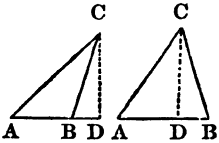
In any obtuse-angled triangle, the square of the side subtending the obtuse angle is greater than the sum of the squares of the other two sides by twice the rectangle of the base and the distance of the perpendicular from the obtuse angle.

Let ABC be a triangle obtuse angled at B, and CD perpendicular to AB; then will the square of AC be greater than the squares of AB, BC, by twice the rectangle of AB, BD. That is, $AC^2=AB^2+BC^2+2AB.BD$. See the 1st figure above.

For $AD^2 = AB^2 + BD^2 + 2AB \cdot BD$ (th. 31).
And $AD^2 + CD^2 = AB^2 + BD^2 + CD^2 + 2AB \cdot BD$ (ax. 2).
But $AD^2 + CD^2 = AC^2$, and $BD^2 + CD^2 = BC^2$ (th. 34).
Therefore $AC^2 = AB^2 + BC^2 + 2AB \cdot BD$. Q. E. D.

THEOREM XXXVII.

In any triangle, the square of the side subtending an acute angle is less than the squares of the base and the other side by twice the rectangle of the base and the distance of the perpendicular from the acute angle.

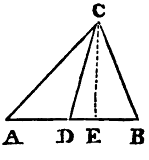


Let ABC be a triangle, having the angle A acute, and CD perpendicular to AB; then will the square of BC be less than the squares of AB, AC, by twice the rectangle of AB, AD. That is, $BC^2 = AB^2 + AC^2 - 2AD \cdot AB$.

For $BD^2 = AD^2 + AB^2 - 2AD \cdot AB$ (th. 32).
And $BD^2 + DC^2 = AD^2 + DC^2 + AB^2 - 2AD \cdot AB$ (ax. 2); therefore $BC^2 = AC^2 + AB^2 - 2AD \cdot AB$ (th. 34). Q. E. D.

THEOREM XXXVIII.

In any triangle, the double of the square of a line drawn from the vertex to the middle of the base, together with double the square of the half base, is equal to the sum of the squares of the other two sides.

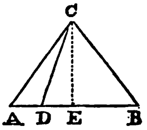


Let ABC be a triangle, and CD the line drawn from the vertex to the middle of the base AB, bisecting it into the two equal parts AD, DB; then will the sum of the squares of AC, CB, be equal to twice the sum of the squares of CD, AD; or $AC^2 + CB^2 = 2CD^2 + 2AD^2$.

For $AC^2 = CD^2 + AD^2 + 2AD \cdot DE$ (th. 36).
And $BC^2 = CD^2 + BD^2 - 2AD \cdot DE$ (th. 37).
Therefore $AC^2 + BC^2 = 2CD^2 + AD^2 + BD^2$.
$= 2CD^2 + 2AD^2$ (ax. 2). Q. E. D.

THEOREM XXXIX.

In an isosceles triangle, the square of a line drawn from the vertex to any point in the base, together with the rectangle of the segments of the base, is equal to the square of one of the equal sides of the triangle.



Let ABC be the isosceles triangle, and CD a line drawn from the vertex to any point D in the base; then will the square of AC be equal to the square of CD, together with the rectangle of AD and DB. That is, $AC^2 = CD^2 + AD \, . \, DB$.

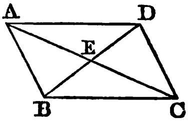
For $AC^2 - CD^2 = AE^2 - DE^2$ (th. 35).

$= AD \, . \, DB$ (th. 33).

Therefore $AC^2 = CD^2 + AD \, . \, DB$ (ax. 2). Q. E. D.

THEOREM XL.

In any parallelogram the two diagonals bisect each other; and the sum of their squares is equal to the sum of the squares of all the four sides of the parallelogram.



Let ABCD be a parallelogram whose diagonals intersect each other in E; then will AE be equal to EC, and BE to ED; and the sum of the squares of AC, BD, will be equal to the sum of the squares of AB, BC, CD, DA. That is,

$AE = EC$, and $BE = ED$,

and $AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2$.

For the triangles AEB, DEC, are equiangular, because they have the opposite angles at E equal (th. 7), and the two lines AC, BD, meeting the parallels AB, DC, make the angle BAE equal to the angle DCE, and the angle ABE equal to the angle CDE, and the side AB equal to the side DC (th. 22); therefore these two triangles are identical, and have their corresponding sides equal (th. 2), viz., $AE = EC$, and $BE = ED$.

Again, since AC is bisected in E, the sum of the squares $AD^2 + DC^2 = 2AE^2 + 2DE^2$ (th. 38).

In like manner, $AB^2 + BC^2 = 2AE^2 + 2BE^2$ or $2DE^2$.

Therefore $AB^2+BC^2+CD^2+DA^2=4AE^2+4DE^2$ (ax. 2).

But because the square of a whole line is equal to four times the square of half the line (cor., th. 31), that is, $AC^2=4AE^2$, and $BD^2=4DE^2$;

Therefore $AB^2+BC^2+CD^2+DA^2=AC^2+BD^2$ (ax. 1). Q. E. D.

Cor. 1. If $AD=DC$, or the parallelogram be a rhombus, then $AD^2=AE^2+ED^2$, $CD^2=DE^2+CE^2$, &c.

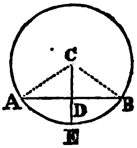
Cor. 2. Hence, and by th. 34, the diagonals of a rhombus intersect at right angles.

BOOK II.

OF THE CIRCLE.

THEOREM I.

If a line, drawn through or from the centre of a circle, bisect a chord, it will be perpendicular to it; or, if it be perpendicular to the chord, it will bisect both the chord and the arc of the chord.



Let AB be any chord in a circle, and CD a line drawn from the centre C to the chord. Then, if the chord be bisected in the point D, CD will be perpendicular to AB.

Draw the two radii CA, CB. Then the two triangles ACD, BCD, having CA equal to CB (def. 45), and CD common, also AD equal to DB (by hyp.); they have all the three sides of the one equal to all the three sides of the other, and so have their angles also equal (th. 5, B. I.). Hence, then, the angle ADC being equal to the angle BDC, these angles are right angles, and the line CD is perpendicular to AB (def. 11).

Again, if CD be perpendicular to AB, then will the chord AB be bisected at the point D, or have AD equal to DB; and the arc AEB bisected in the point E, or have AE equal EB.

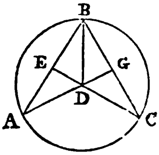
For, having drawn CA, CB, as before; then, in the triangle ABC, because the side CA is equal to the side CB, their opposite angles A and B are also equal (th. 3). Hence, then, in the two triangles ACD, BCD, the angle A is equal to the angle B, and the angles at D are equal (def. 11); therefore their third angles are also equal (corol. 1, th. 18, B. I.). And having the side CD common, they have also the side AD equal to the side DB (th. 2, B. I.).

Also, since the angle ACE is equal to the angle BCE, the arc AE, which measures the former (def. 58), is equal to the arc BE, which measures the latter, since equal angles must have equal measures.

Corol. Hence a line bisecting any chord at right angles passes through the centre of the circle.

THEOREM II.

If more than two equal lines can be drawn from any point within a circle to the circumference, that point will be the centre.



Let ABC be a circle, and D a point within it; then, if any three lines, DA, DB, DC, drawn from the point D to the circumference, be equal to each other, the point D will be the centre.

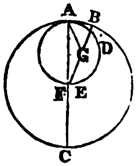
Draw the chords AB, BC, which let be bisected in the points E, G, and join DE, DG.

Then the two triangles DAE, DBE, have the side DA equal to the side DB by supposition, and the side AE equal to the side EB by hypothesis, also the side DE common; therefore these two triangles are identical, and have the angles at E equal to each other (th. 5); consequently, DE is perpendicular to the middle of the chord AB (def. 11), and therefore passes through the centre of the circle (corol., th. 1).

In like manner, it may be shown that DG passes through the centre. Consequently, the point D is the centre of the circle, and the three equal lines DA, DB, DC, are radii. Q. E. D.

THEOREM III.

If two circles, placed one within another, touch, the centres of the circles and the point of contact will be all in the same right line.



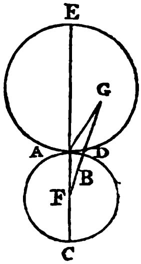
Let the two circles ABC, ADE, touch one another internally in the point A; then will the point A and the centres of those circles be all in the same right line.

Let F be the centre of the circle ABC, through which draw the diameter AFC. Then, if the centre of the other circle can be out of this line AC, let it be supposed in some other point, as G, through which draw the line FG, cutting the two circles in B and D.

Now, in the triangle AFG, the sum of the two sides FG, GA, is greater than the side AF (th. 10, B. I.), or greater than its equal radius FB. From each of these take away the common part FG, and the remainder GA will be greater than the remainder GB. But the point G being supposed the centre of the inner circle, its two radii, GA, GD, are equal to each other; consequently, GD will also be greater than GB. But ADE being the inner circle, GD is necessarily less than GB. So that GD is both greater and less than GB, which is absurd. To get rid of this absurdity, we must abandon the supposition that produced it, which was that G might be out of the line AFC. Consequently, the centre G *can not* be out of the line AFC. Q. E. D.

THEOREM IV.

If two circles touch one another externally, the centres of the circles and the point of contact will be all in the same right line.



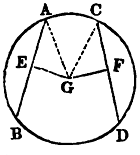
Let the two circles ABC, ADE, touch one another externally at the point A; then will the point of contact A and the centres of the two circles be all in the same right line.

Let F be the centre of the circle ABC, through which draw the diameter AFC, and produce it to the other circle at E. Then, if the centre of the other circle ADE can be out of the line FE, let it, if possible, be supposed in some other point, as G, and draw the lines AG, FBDG, cutting the two circles in B and D.

Then, in the triangle AFG, the sum of the two sides AF, AG, is greater than the third side FG (th. 10, B. I.). But F and G being the centres of the two circles, the two radii GA, GD, are equal, as are also the two radii AF, FB. Hence the sum of GA, AF, is equal to the sum of GD, BF; and therefore this latter sum also, GD, BF, is greater than GF, which is absurd. Consequently, the centre G can not be out of the line EF. Q. E. D.

THEOREM V.

Any chords in a circle which are equally distant from the centre are equal to each other; or, if they be equal to each other, they will be equally distant from the centre.



Let AB, CD, be any two chords at equal distances from the centre, G; then will these two chords, AB, CD, be equal to each other.

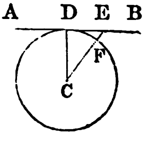
Draw the two radii GA, GC, and the two perpendiculars GE, GF, which are the equal distances from the centre, G. Then, the two right-angled triangles, GAE, GCF, having the side GA equal the side GC, and the side GE equal the side GF, and the angle at E equal to the angle at F, therefore those two triangles are identical (Cor. 2, th. 34, B. I.), and have the line AE equal to the line CF. But AB is the double of AE, and CD is the double of CF (th. 1); therefore AB is equal to CD (by ax. 6). Q. E. D.

Again, if the chord AB be equal to the chord CD, then will their distances from the centre, GE, GF, also be equal to each other.

For, since AB is equal CD by supposition, the half AE is equal the half CF. Also, the radii GA, GC, being equal, as well as the right angles E and F, therefore the third sides are equal (Cor. 2, th. 34), or the distance GE equal the distance GF. Q. E. D.

THEOREM VI.

A line perpendicular to a radius at its extremity is a tangent to the circle.



Let the line ADB be perpendicular to the radius CD of a circle; then shall AB touch the circle in the point D only.

From any other point, E, in the line AB, draw CFB to the centre, cutting the circle in F.

Then, because the angle D of the triangle CDE is a right angle, the angle at E is acute (Cor. 3, th. 18, B. I.), and, consequently, less than the angle D. But the greater side is always opposite to the greater angle (th. 9, B. I.), therefore the side CE is greater than the side CD, or greater than its equal CF. Hence the point E is without the circle; and the same for every other point in the line AB. Consequently the whole line is without the circle, and meets it in the point D only.

THEOREM VII.

When a line is a tangent to a circle, a radius drawn to the point of contact is perpendicular to the tangent.

Let the line AB touch the circumference of a circle at the point D; then will the radius CD be perpendicular to the tangent AB. [See the last figure.]

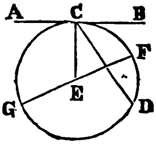
For, the line AB being wholly without the circumference except at the point D, every other line, as CE, drawn from the centre C to the line AB, must pass out of the circle to arrive at this line. The line CD is therefore the shortest that can be drawn from the point C to the line AB, and, consequently (th. 21, B. I.), it is perpendicular to that line.

Corol. Hence, conversely, a line drawn perpendicular to a tangent at the point of contact passes through the centre of the circle.

THEOREM VIII.

The angle formed by a tangent and chord is measured by half the arc of that chord.

Let AB be a tangent to a circle, and CD a chord drawn from the point of contact, C; then is the angle BCD measured by half the arc CFD, and the angle ACD measured by half the arc CGD.



Draw the radius EC to the point of contact, and the radius EF perpendicular to the chord at H.

Then the radius EF, being perpendicular to the chord CD, bisects the arc CFD (th. 1). Therefore CF is half the arc CFD.

In the triangle CEH, the angle H being a right one, the sum of the two remaining angles E and C is equal to a right angle (Cor. 3, th. 17, B. I.), which is equal to the angle

BCE, because the radius CE is perpendicular to the tangent. From each of these equals take away the common part or angle C, and there remains the angle E equal to the angle BCD. But the angle E is measured by the arc CF (def. 57), which is the half of CFD; therefore the equal angle BCD must also have the same measure, namely, half the arc CFD of the chord CD.

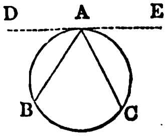
Again, the line GEF, being perpendicular to the chord CD, bisects the arc CGD (th. 1). Therefore CG is half the arc CGD. Now, since the line CE, meeting FG, makes the sum of the two angles at E equal to two right angles (th. 6, B. I.), and the line CD makes with AB the sum of the two angles at C equal to two right angles; if from these two equal sums there be taken away the parts or angles CEH and BCH, which have been proved equal, there remains the angle CEG equal to the angle ACH. But the former of these, CEG, being an angle at the centre, is measured by the arc CG (def. 58); consequently, the equal angle ACD must also have the same measure CG, which is half the arc CGD of the chord CD. Q. E. D.

Corol. 1. The sum of the two right angles is measured by half the circumference. For the two angles BCD, ACD, which make up two right angles, are measured by the arcs CF, CG, which make up half the circumference, FG being a diameter.

Corol. 2. Hence, also, one right angle must have for its measure a quarter of the circumference, or 90 degrees.

THEOREM IX.

An angle at the circumference of a circle is measured by half the arc that subtends it.

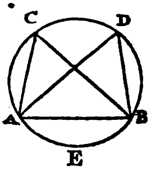


Let BAC be an angle at the circumference: it has for its measure half the arc BC, which subtends it.

For, suppose the tangent DE to pass through the point of contact A; then, the angle DAC being measured by half the arc ABC, and the angle DAB by half the arc AB (th. 8), it follows, by equal subtraction, that the difference, or angle BAC, must be measured by half the arc BC, which it stands upon. Q. E. D.

THEOREM X.

All angles in the same segment of a circle, or standing on the same arc, are equal to each other.

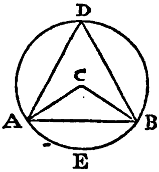


Let C and D be two angles in the same segment ACDB, or, which is the same thing, standing on the supplemental arc AEB; then will the angle C be equal to the angle D.

For each of these angles is measured by half the arc AEB; and thus, having equal measures, they are equal to each other (ax. 11).

THEOREM XI.

An angle at the centre of a circle is double the angle at the circumference, when both stand on the same arc.



Let C be an angle at the centre C, and D an angle at the circumference, both standing on the same arc or same chord AB; then will the angle C be double of the angle D, or the angle D equal to half the angle C.

For the angle at the centre C is measured by the whole arc AEB (def. 58), and the angle at the circumference D is measured by half the same arc AEB (th. 9); therefore the angle D is only half the angle C, or the angle C double the angle D.

THEOREM XII.

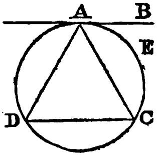
An angle in a semicircle is a right angle.

If ABC or ADC be a semicircle, then any angle D in that semicircle is a right angle.

For the angle D, at the circumference, is measured by half the arc ABC (th. 9), that is, by a quadrant of the circumference. But a quadrant is the measure of a right angle (Cor. 4, th. 6; or Cor. 2, th. 8). Therefore the angle D is a right angle.

THEOREM XIII.

The angle formed by a tangent to a circle, and a chord drawn from the point of contact, is equal to the angle in the alternate segment.

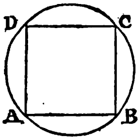


If AB be a tangent, and AC a chord, and D any angle in the alternate segment ADC, then will the angle D be equal to the angle BAC made by the tangent and chord of the arc AEC.

For the angle D, at the circumference, is measured by half the arc AEC (th. 9); and the angle BAC, made by the tangent and chord, is also measured by the same half arc AEC (th. 8); therefore these two angles are equal (ax. 11).

THEOREM XIV.

The sum of any two opposite angles of a quadrangle inscribed in a circle is equal to two right angles.



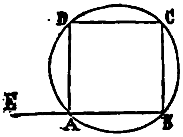
Let ABCD be any quadrilateral inscribed in a circle; then shall the sum of the two opposite angles A and C, or B and D, be equal to two right angles.

For the angle A is measured by half the arc DCB, which it stands upon, and the angle C by half the arc DAB (th. 9); therefore the sum of the two angles A and C is measured by half the sum of these two arcs, that is, by half the circumference. But half the circumference is the measure of two right angles (Cor. 4, th. 6, B. I.); therefore the sum of the two opposite angles A and C is equal to two right angles. In like manner it is shown that the sum of the other two opposite angles, D and B, is equal to two right angles. Q. E. D

THEOREM XV.

If any side of a quadrangle inscribed in a circle be produced out, the exterior angle will be equal to the interior opposite angle.

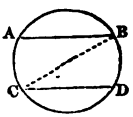
If the side AB, of the quadrilateral ABCD, inscribed in a circle, be produced to E, the exterior angle DAE will be equal to the interior opposite angle C.



For the sum of the two adjacent angles DAE and DAB is equal to two right angles (th. 6), and the sum of the two opposite angles C and DAB is also equal to two right angles (th. 14); therefore the former sum of the two angles DAE and DAB is equal to the latter sum of the two C and DAB (ax. 1). From each of these equals taking away the common angle DAB, there remains the angle DAE equal the angle C. Q. E. D.

THEOREM XVI.

Any two parallel chords intercept equal arcs.

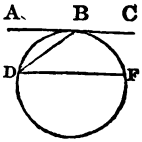


Let the two chords AB, CD, be parallel; then will the arcs AC, BD, be equal; or AC =DB.

Draw the line BC. Then, because the lines AB, CD, are parallel, the alternate angles B and C are equal (th. 13, B. I.). But the angle at the circumference B is measured by half the arc AC (th. 9), and the other equal angle at the circumference C is measured by half the arc BD; therefore the halves of the arcs AC, BD, and, consequently, the arcs themselves, are also equal. Q. E. D.

THEOREM XVII.

When a tangent and chord are parallel to each other, they intercept equal arcs.



Let the tangent ABC be parallel to the chord DF; then are the arcs BD, BF, equal; that is, BD=BF.

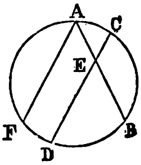
Draw the chord BD. Then, because the lines AB, DF, are parallel, the alternate angles D and B are equal (th. 13, B. I.). But the angle B, formed by a tangent and chord, is measured by half the arc BD (th. 8), and the other angle at the circumference D is measured by half the arc BF (th. 9); therefore the arcs BD, BF are equal. Q. E. D.

THEOREM XVIII.

The angle formed within a circle by the intersection of two chords is measured by half the sum of the two intercepted arcs.

Let the two chords AB, CD intersect at the point E; then the angle AEC or DEB is measured by half the sum of the two arcs AC, DB.

Draw the chord AF parallel to CD. Then, because the lines AF, CD are parallel, and AB cuts them, the angles on the same side A and DEB are equal (th. 15, B. I.). But the angle at the circumference A is measured by half the arc BF, or of the sum of FD and DB (th. 9); therefore the angle E is also measured by half the sum of FD and DB.



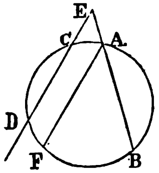
Again, because the chords AF, CD are parallel, the arcs AC, FD are equal (th. 16); therefore the sum of the two arcs AC, DB is equal to the sum of the two FD, DB; and, consequently, the angle E, which is measured by half the latter sum, is also measured by half the former. Q. E. D.

THEOREM XIX.

The angle formed out of a circle by two secants is measured by half the difference of the intercepted arcs.

Let the angle E be formed by two secants EAB and ECD; this angle is measured by half the difference of the two arcs AC, DB, intercepted by the two secants.

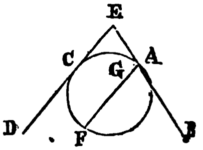
Draw the chord AF parallel to CD. Then, because the lines AF, CD are parallel, and AB cuts them, the angles on the same side A and BED are equal (th. 15, B. I.). But the angle A, at the circumference, is measured by half the arc BF (th. 9), or of the difference of DF and DB; therefore the equal angle E is also measured by half the difference of DF, DB.



Again, because the chords AF, CD are parallel, the arcs AC, FD are equal (th. 16); therefore the difference of the two arcs AC, DB is equal to the difference of the two DF, DB. Consequently, the angle E, which is measured by half the latter difference, is also measured by half the former. Q. E. D.

THEOREM XX.

The angle formed by two tangents is measured by half the difference of the two intercepted arcs.



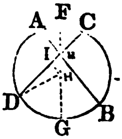
Let EB, ED, be two tangents to a circle at the points A, C; then the angle E is measured by half the difference of the two arcs CFA, CGA.

Draw the chord AF parallel to ED. Then, because the lines AF, ED are parallel, and EB meets them, the angles on the same side A and E are equal (th. 15). But the angle A, formed by the chord AF and tangent AB, is measured by half the arc AF (th. 8); therefore the equal angle E is also measured by half the same arc AF, or half the difference of the arcs CFA and CF, or CGA (th. 17).

Corol. In like manner it is proved that the angle formed by a tangent and a secant is measured by half the difference of the two intercepted arcs.

THEOREM XXI.

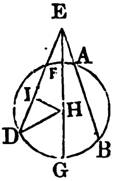
When two lines, meeting a circle each in two points, cut one another, either within it or without it, the rectangle of the parts of the one is equal to the rectangle of the parts of the other, the parts of each being measured from the point of meeting to the two intersections with the circumference.



Let the two lines AB, CD meet each other in E; then the rectangle of AE, EB will be equal to the rectangle of CE, ED. Or, AE . EB=CE . ED.

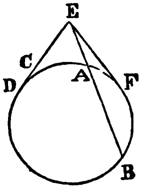
For through the point E draw the diameter FG; also, from the centre H draw the radius DH, and draw HI perpendicular to CD.

Then, since DEH is a triangle, and the perpendicular HI bisects the chord CD (th. 1), the line CE is equal to the difference of the segments DI, EI, the sum of them being DE.



Also, because H is the centre of the circle, and the radii DH, FH, GH, are all equal, the line EG is equal to the sum of the sides DH, HE, and EF is equal to their difference.

But the rectangle of the sum and difference of the two sides of a triangle is equal to the rectangle of the sum and difference of the segments of the base (th. 35, B. I.); therefore the rectangle of FE, EG is equal to the rectangle of CE, ED. In like manner it is proved that the same rectangle of FE, EG is equal to the rectangle of AE, EB. Consequently, the rectangle of AE, EB is also equal to the rectangle of CE, ED (ax. 1). Q. E. D.

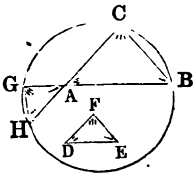


Corol. 1. When one of the lines in the second case, as DE, by revolving about the point E, comes into the position of the tangent EC or ED, the two points C and D running into one, then the rectangle of CE, ED becomes the square of CE, because CE and DE are then equal. Consequently, the rectangle of the parts of the secant $AE . EB$ is equal to the square of the tangent CE^2.

Corol. 2. Hence both the tangents EC, EF, drawn from the same point E, are equal, since the square of each is equal to the same rectangle or quantity $AE . EB$.

THEOREM XXII.

In equiangular triangles, the rectangles of the corresponding or like sides, taken alternately, are equal.



Let ABC, DEF be two equiangular triangles having the angle A = the angle D, the angle B = the angle E, and the angle C = the angle F; also the like sides AB, DE, and AC, DF, being those opposite the equal angles; then will the rectangle of AB, DF be equal to the rectangle of AC, DE.

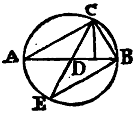
In BA produced take AG equal to DF, and through the three points B, C, G conceive a circle BCGH to be described, meeting CA produced at H, and join GH.

Then the angle G is equal to the angle C on the same arc BH, and the angle H equal to the angle B on the same arc CG (th. 10); also the opposite angles at A are equal (th. 7, B. I.); therefore the triangle AGH is equiangular to the triangle ACB, and consequently to the triangle DFE also. But the two like sides AG, DF are also equal by supposition, consequently the two triangles AGH, DFE are identical (th. 2, B. I.), having the two sides AG, AH equal to the two DF, DE, each to each.

But the rectangle GA . AB is equal to the rectangle HA . AC (th. 21); consequently the rectangle DF . AB is equal to the rectangle DE . AC. Q. E. D.

THEOREM XXIII.

The rectangle of the two sides of any triangle, is equal to the rectangle of the perpendicular on the third side and the diameter of the circumscribing circle.



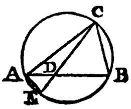
Let CD be the perpendicular, and CE the diameter of the circle about the triangle ABC; then the rectangle CA . CB is $=$ the rectangle CD . CE.

For, join BE; then, in the two triangles ACD, ECB, the angles A and E are equal, standing on the same arc BC (th. 10); also the right angle D is equal to the angle B, which is also a right angle, being in a semicircle (th. 12); therefore these two triangles have also their third angles equal, and are equiangular. Hence AC, CE, and CD, CB, being like sides, subtending the equal angles, the rectangle AC . CB, of the first and last of them is equal to the rectangle CE . CD of the other two (th. 22).

THEOREM XXIV.

The square of a line bisecting any angle of a triangle, together with the rectangle of the two segments of the opposite side, is equal to the rectangle of the two other sides including the bisected angle.

Let CD bisect the angle C of the triangle ABC; then the square CD^2+ the rectangle AD . DB is $=$ the rectangle AC . CB.

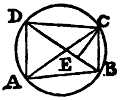


For, let CD be produced to meet the circumscribing circle at E, and join AE.

Then the two triangles AEC, BCD are equiangular; for the angles at C are equal by supposition, and the angles B and E are equal, standing on the same arc AC (th. 10); consequently the third angles at A and D are equal (Cor. 1, th. 18, B. I.); also AC, CD, and CE, CB, are like or corresponding sides, being opposite to equal angles; therefore the rectangle AC . CB is=the rectangle CD . CE (th. 22). But the latter rectangle CD . CE is$=CD^2+$the rectangle CD . DE (th. 30, B. I.); therefore the former rectangle AC . CB is also$=CD^2+$CD . DE, or equal to CD^2+AD . DB, since CD . DE is=AD . DB (th. 21). Q. E. D.

THEOREM XXV.

The rectangle of the two diagonals of any quadrangle inscribed in a circle is equal to the sum of the two rectangles of the opposite sides.



Let ABCD be any quadrilateral inscribed in a circle, and AC, BD its two diagonals; then the rectangle AC . BD is=the rectangle AB . DC+the rectangle AD . BC.

For, let CE be drawn, making the angle BCE equal to the angle DCA. Then the two triangles ACD, BCE are equiangular; for the angles A and B are equal, standing on the same arc CD; and the angles DCA, BCE, are equal by supposition; consequently, the third angles ADC, BEC are also equal; also AC, BC, and AD, BE, are like or corresponding sides, being opposite to the equal angles; therefore the rectangle AC . BE is=the rectangle AD . BC (th. 22).

Again, the two triangles ABC, DEC are equiangular; for the angles BAC, BDC are equal, standing on the same arc BC; and the angle DCE is equal to the angle BCA, by adding the common angle ACE to the two equal angles DCA, BCE; therefore the third angles E and ABC are also equal; but AC, DC, and AB, DE, are the like sides; therefore the rectangle AC . DE is=the rectangle AB . DC (th. 22).

Hence, by equal additions, the sum of the rectangles AC .

BE+AC . DE is=AD . BC+AB . DC. But the former sum of the rectangles AC . BE+AC . DE is=the rectangle AC . BD (th. 30, B. I.); therefore the same rectangle AC . BD is equal to the latter sum, the rectangle AD . BC+the rectangle AB . DC (ax. 1). Q. E. D.

Corol. Hence, if ABD be an equilateral triangle, and C any point in the arc BCD of the circumscribing circle, we have AC=BC+DC. For AC . BD being = AD . BC+AB . DC; dividing by BD=AB=AD, there results AC=BC+DC.

BOOK III.

OF RATIOS AND PROPORTIONS.

DEFINITIONS.

DEF. 78. RATIO is the proportion or relation which one magnitude bears to another magnitude of the same kind with respect to quantity.

Note. The measure or quantity of a ratio is conceived by considering what part or parts the leading quantity, called the antecedent, is of the other, called the consequent; or what part or parts the number expressing the quantity of the former is of the number denoting in like manner the latter. So the ratio of a quantity expressed by the number 2 to a like quantity expressed by the number 6, is denoted by 2 divided by 6, or $\frac{2}{6}$ or $\frac{1}{3}$: the number 2 being 3 times contained in 6, or the third part of it. In like manner, the ratio of the quantity 3 to 6 is measured by $\frac{3}{6}$ or $\frac{1}{2}$; the ratio of 4 to 6 is $\frac{4}{6}$ or $\frac{2}{3}$; that of 6 to 4 is $\frac{6}{4}$ or $\frac{3}{2}$, &c.

79. Proportion is an equality of ratios. Thus,

80. Three quantities are said to be proportional when the ratio of the first to the second is equal to the ratio of the second to the third. As of the three quantities A (2), B (4), C (8), where $\frac{2}{4}=\frac{4}{8}=\frac{1}{2}$, both the same ratio.

81. Four quantities are said to be proportional when the ratio of the first to the second is the same as the ratio of the third to the fourth. As of the four A (4), B (2), C (10), D (5), where $\frac{4}{2}=\frac{10}{5}=2$, both the same ratio.

Note. To denote that four quantities, A, B, C, D, are proportional, they are usually stated or placed thus: A : B :: C : D, and read thus: A is to B as C is to D. But when three quantities are proportional, the middle one is repeated, and they are written thus: A : B :: B : C.

The proportionality of quantities may also be expressed very generally by the equality of fractions.

Thus, if $\frac{A}{B}=\frac{C}{D}$, then A : B :: C : D; also B : A :: D : C, A : C :: B : D, and C : A :: D : B.

82. Of three proportional quantities, the middle one is said to be a mean proportional between the other two, and the last a third proportional to the first and second.

83. Of four proportional quantities, the last is said to be a fourth proportional to the other three, taken in order.

84. Quantities are said to be continually proportional, or in continued proportion, when the ratio is the same between every two adjacent terms, viz., when the first is to the second, as the second to the third, as the third to the fourth, as the fourth to the fifth, and so on, all in the same common ratio, as in the quantities, 1, 2, 4, 8, 16, &c., where the common ratio is equal to 2.

85. Of any number of quantities, A, B, C, D, the ratio of the first, A, to the last, D, is said to be compounded of the ratios of the first to the second, of the second to the third, and so on to the last.

86. Inverse ratio is when the antecedent is made the consequent, and the consequent the antecedent. Thus, if 1 : 2 :: 3 : 6, then inversely, 2 : 1 :: 6 : 3.

87. Alternate proportion is when antecedent is compared with antecedent, and consequent with consequent. As, if 1 : 2 :: 3 : 6, then, by alternation or permutation, it will be 1 : 3 :: 2 : 6.

88. Compound ratio is when the sum of the antecedent and consequent is compared either with the consequent or with the antecedent. Thus, if 1 : 2 :: 3 : 6, then, by composition, $1+2:1::3+6:3$, and $1+2:2::3+6:6$.

89. Divided ratio is when the difference of the antecedent and consequent is compared either with the antecedent or with the consequent. Thus, if 1 : 2 :: 3 : 6, then, by division, $2-1:1::6-3:3$, and $2-1:2::6-3:6$.

Note.—The term Divided, or Division, here means subtracting or parting, being used in the sense opposed to compounding or adding, in Def. 88.

THEOREM I.

Equimultiples of any two quantities have the same ratio as the quantities themselves.

Let A and B be any two quantities, and mA, mB, any equimultiples of them, m being any number whatever; then will mA and mB have the same ratio as A and B, or A : B :: mA : mB.

For $\frac{mB}{mA}=\frac{B}{A}$, the same ratio.

Corol. Hence like parts of quantities have the same ratio as the wholes, because the wholes are equimultiples of the like parts, or A and B are like parts of mA and mB.

THEOREM II.

If four quantities of the same kind be proportionals, they will be in proportion by alternation or permutation, or the antecedents will have the same ratio as the consequents.

Let A : B :: mA : mB; then will A : mA :: B : mB.

For $\frac{mA}{A}=\frac{m}{1}$, and $\frac{mB}{B}=\frac{m}{1}$, both the same ratio.

THEOREM III.

If four quantities be proportional, they will be in proportion by inversion or inversely.

Let A : B :: mA : mB; then will B : A :: mB : mA.

For $\frac{mA}{mB}=\frac{A}{B}$, both the same ratio.

Otherwise. Let A : B :: C : D; then shall B : A :: D : C.

For, let $\frac{A}{B}=\frac{C}{D}=r$; then $A=Br$, and $C=Dr$; therefore $B=\frac{A}{r}$, and $D=\frac{C}{r}$. Hence $\frac{B}{A}=\frac{1}{r}$, and $\frac{D}{C}=\frac{1}{r}$. Whence it is evident that $\frac{B}{A}=\frac{D}{C}$ (ax. 1), or B : A :: D : C.

In a similar manner may most of the other theorems be demonstrated.

THEOREM IV.

If four quantities be proportional, they will be in proportion by composition and division.

Let $A : B :: mA : mB$.

Then will $B \pm A : A :: mB \pm mA : mA$,

and $B \pm A : B :: mB \pm mA : mB$.

For $\frac{mA}{mB \pm mA} = \frac{A}{B \pm A}$; and $\frac{mB}{mB \pm mA} = \frac{B}{B \pm A}$.

Corol. It appears from hence that the sum of the greatest and least of four proportional quantities of the same kind exceeds the sum of the other two. For, since $A : A+B :: mA :: mA+mB$ - - where A is the least, and $mA+mB$ the greatest; then $(m+1).A+mB$, the sum of the greatest and least, exceeds $(m+1).A+B$, the sum of the two other quantities.

THEOREM V.

If of four proportional quantities there be taken any equimultiples whatever of the two antecedents, and any equimultiples whatever of the two consequents, the quantities resulting will still be proportional.

Let $A : B :: mA : mB$; also, let pA and pmA be any equimultiples of the two antecedents, and qB and qmB any equimultiples of the two consequents; then will - - - - $pA : qB :: pmA : qmB$.

For $\frac{qmB}{pmA} = \frac{qB}{pA}$, both the same ratio.

THEOREM VI.

If there be four proportional quantities, and the two consequents be either augmented or diminished by quantities that have the same ratio as the respective antecedents, the results and the antecedents will still be proportionals.

Let $A : B :: mA : mB$, and nA and nmA any two quantities having the same ratio as the two antecedents; then will $A : B \pm nA :: mA : mB \pm nmA$.

For $\frac{mB \pm nmA}{mA} = \frac{B \pm nA}{A}$, both the same ratio.

THEOREM VII.

If any number of quantities be proportional, then any one of the antecedents will be to its consequent as the sum of all the antecedents is to the sum of all the consequents.

Let $A : B :: mA : mB :: nA : nB$, &c.; then will $A : B :: A + mA + nA : B + mB + nB$, &c.

For $\frac{B+mB+nB}{A+mA+nA} = \frac{(1+m+n)B}{(1+m+n)A} = \frac{B}{A}$, the same ratio.

THEOREM VIII.

If a whole magnitude be to a whole as a part taken from the first is to a part taken from the other, then the remainder will be to the remainder as the whole to the whole.

Let $A : B :: \frac{m}{n}A : \frac{m}{n}B$;

Then will $A : B :: A - \frac{m}{n}A : B - \frac{m}{n}B$.

For $\frac{B - \frac{m}{n}B}{A - \frac{m}{n}A} = \frac{B}{A}$, both the same ratio.

THEOREM IX.

If any quantities be proportional, their squares, or cubes, or any like powers or roots of them will also be proportional.

Let $A : B :: mA : mB$; then will $A^n : B^n :: m^nA^n : m^nB^n$.

For $\frac{m^nB^n}{m^nA^n} = \frac{B^n}{A^n}$, both the same ratio.

See also theorem 8.

THEOREM X.

If there be two sets of proportionals, then the products or rectangles of the corresponding terms will also be proportional.

Let $A : B :: mA : mB$,
and $C : D :: nC : nD$;
then will $AC : BD :: mnAC : mnBD$.

For $\frac{mn\text{BD}}{mn\text{AC}}=\frac{\text{BD}}{\text{AC}}$, both the same ratio.

THEOREM XI.

If four quantities be proportional, the rectangle or product of the two extremes will be equal to the rectangle or product of the two means. And the converse.

Let $\text{A}:\text{B}::m\text{A}:m\text{B}$;
then is $\text{A}\times m\text{B}=\text{B}\times m\text{A}=m\text{AB}$, as is evident.

THEOREM XII.

If three quantities be continued proportionals, the rectangle or product of the two extremes will be equal to the square of the mean. And the converse.

Let A, $m\text{A}$, $m^2\text{A}$, be three proportionals;
or $\text{A}:m\text{A}::m\text{A}:m^2\text{A}$;
then is $\text{A}\times m^2\text{A}=m^2\text{A}^2$, as is evident.

THEOREM XIII.

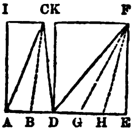
If any number of quantities be continued proportionals, the ratio of the first to the third will be duplicate or the square of the ratio of the first and second; and the ratio of the first and fourth will be triplicate or the cube of that of the first and second; and so on.

Let A, $m\text{A}$, $m^2\text{A}$, $m^3\text{A}$, &c., be proportionals;
then is $\frac{\text{A}}{m\text{A}}=\frac{1}{m}$; but $\frac{\text{A}}{m^2\text{A}}=\frac{1}{m^2}$; and $\frac{\text{A}}{m^3\text{A}}=\frac{1}{m^3}$, &c.

BOOK IV.

THEOREM I.

Triangles, and also parallelograms, having equal altitudes, are to each other as their bases.



Let the two triangles ADC, DEF, have the same altitude, or be between the same parallels AE, IF; then is the surface of the triangle ADC to the surface of the triangle DEF as the base AD is to the base DE; or AD : DE :: the triangle ADC : the triangle DEF.

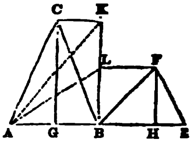
For, let the base AD be to the base DE as any one number m (2) to any other number n (3), and divide the respective bases into those parts, AB, BD, DG, GH, HE, all equal to one another, and from the points of division draw the lines BC, GF, HF to the vertices C and F; then will these lines divide the triangles ADC, DEF, into the same number of parts as their bases, each equal to the triangle ABC, because those triangular parts have equal bases and altitude (Cor. 2, th. 25, B. I.); namely, the triangle ABC equal to each of the triangles BDC, DFG, GFH, HFE. So that the triangle ADC is to the triangle DFE as the number of parts m (2) of the former to the number n (3) of the latter; that is, as the base AD to the base DE.

In like manner, the parallelogram ADKI is to the parallelogram DEFK as the base AD is to the base DE, each of these having the same ratio as the number of their parts, m to n. Q. E. D.

THEOREM II.

Triangles, and also parallelograms, having equal bases, are to each other as their altitudes.

Let ABC, BEF, be two triangles having the equal bases



AB, BE, and whose altitudes are the perpendiculars CG, FH; then will the triangle ABC : the triangle BEF :: CG : FH.

For, let BK be perpendicular to AB, and equal to CG; in which let there be taken BL=FH, drawing AK and AL.

Then, triangles of equal bases and heights being equal (Cor. 2, th. 25, B. I.), the triangle ABK is=ABC, and the triangle ABL=BEF. But, considering now ABK, ABL, as two triangles on the bases BK, BL, and having the same altitude AB, these will be as their bases (th. 1), namely, the triangle ABK : the triangle ABL :: BK : BL.

But the triangle ABK=ABC, and the triangle ABL=BEF; also BK=CG, and BL=FH.

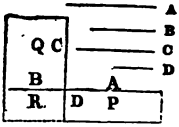
Therefore the triangle ABC : triangle BEF :: CG : FH.

And since parallelograms are the doubles of these triangles, having the same bases and altitudes, they will likewise have to each other the same ratio as their altitudes. Q. E. D.

Corol. Since, by this theorem, triangles and parallelograms, when their bases are equal, are to each other as their altitudes; and, by the foregoing one, when their altitudes are equal, they are to each other as their bases; therefore, universally, when neither are equal, they are to each other in the compound ratio, or as the rectangle or product of their bases and altitudes.

THEOREM III.

If four lines be proportional, the rectangle of the extremes will be equal to the rectangle of the means; and, conversely, if the rectangle of the extremes of four lines be equal to the rectangle of the means, the four lines, taken alternately, will be proportional.



Let the four lines A, B, C, D be proportionals, or A : B :: C : D; then will the rectangle of A and D be equal to the rectangle of B and C; or the rectangle A.D =B.C.

For, let the four lines be placed with their four extremities meeting in a common point, forming at that point four right angles; and draw lines parallel to them to complete the rectangles P, Q, R, where P is the rectangle of A and D, Q the rectangle of B and C, and R the rectangle of B and D.

Then the rectangles P and R, being between the same parallels, are to each other as their bases A and B (th. 1); and the rectangles Q and R, being between the same parallels, are to each other as their bases C and D. But the ratio of A to B is the same as the ratio of C to D, by hypothesis; therefore the ratio of P to R is the same as the ratio of Q to R; and, consequently, the rectangles P and Q are equal. Q. E. D.

Again, if the rectangle of A and D be equal to the rectangle of B and C, these lines will be proportional, or A : B :: C : D.

For, the rectangles being placed the same as before, then, because parallelograms between the same parallels are to one another as their bases, the rectangle P : R :: A : B, and Q : R :: C : D. But, as P and Q are equal, by supposition, they have the same ratio to R; that is, the ratio of A to B is equal to the ratio of C to D, or A : B :: C : D. Q. E. D.

Corol. 1. When the two means, namely, the second and third terms, are equal, their rectangle becomes a square of the second term, which supplies the place of both the second and third. And hence it follows that when three lines are proportionals, the rectangle of the two extremes is equal to the square of the mean; and, conversely, if the rectangle of the extremes be equal to the square of the mean, the three lines are proportionals.

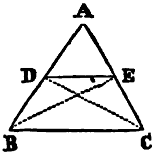
Corol. 2. Since it appears by the rules of proportion in arithmetic and algebra that when four quantities are proportional, the product of the extremes is equal to the product of the two means; and, by this theorem, the rectangle of the extremes is equal to the rectangle of the two means; it follows that the area or space of a rectangle is represented or expressed by the product of its length and breadth multiplied together. And, in general, a rectangle in geometry is similar to the product of the measures of its two dimensions of length and breadth, or base and height. Also, a square is similar to, or represented by, the measure of its side multiplied by itself. So that what is shown of such products is to be understood of the squares and rectangles.

Corol. 3. Since the same reasoning as in this theorem holds for any parallelograms whatever, as well as for the rectangles, the same property belongs to all kinds of parallelograms, having equal angles, and also to triangles, which are the halves of parallelograms; namely, that if the sides about the equal angles of parallelograms or triangles be reciprocally proportional, the parallelograms or triangles will be equal; and, conversely, if the parallelograms or triangles be equal, their sides about the equal angles will be reciprocally proportional.

Corol. 4. Parallelograms or triangles, having an angle in each equal, are in proportion to each other as the rectangles of the sides which are about these equal angles.

THEOREM IV.

If a line be drawn in a triangle parallel to one of its sides, it will cut the other two sides proportionally.



Let DE be parallel to the side BC of the triangle ABC; then will AD : DB :: AE : EC.

For, draw BE and CD; then the triangles DBE, DCE, are equal to each other, because they have the same base DE, and are between the same parallels DE, BC (th. 25, B. I.). But the two triangles ADE, BDE, on the bases AD, DB, have the same altitude; and the two triangles ADE, CDE, on the bases AE, EC, have also the same altitude; and because triangles of the same altitude are to each other as their bases, therefore

the triangle ADE : BDE :: AD : DB,
and triangle ADE : CDE :: AE : EC.

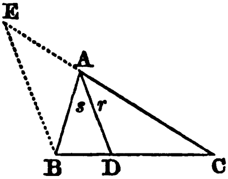
But BDE is = CDE, and equals must have to equals the same ratio; therefore AD : DB :: AE : EC. Q. E. D.

Corol. Hence, also, the whole lines AB, AC are proportional to their corresponding proportional segments (Cor., th. 1, B. III.),

viz., AB : AC :: AD : AE,
and AB : AC :: BD : CE.

THEOREM V.

A line which bisects any angle of a triangle divides the opposite side into two segments, which are proportional to the two other adjacent sides.

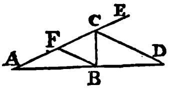


Let the angle CAB of the triangle ABC be bisected by the line AD, making the angle r equal to the angle s; then will the segment CD be to the segment DB as the side AC is to the side AB; or CD : DB :: AC : AB.

For, let BE be parallel to AD, meeting CA produced at E; then, because the line BA cuts the two parallels AD, BE, it makes the angle ABE equal to the alternate angle s (th. 13, B. I.), and therefore also equal to the angle r, which is equal to s by the supposition. Again, because the line CE cuts the two parallels DA, BE, it makes the angle E equal to the angle r on the same side of it (th. 15, B. I.). Hence, in the triangle BAE, the angles B and E, being each equal to the angle r, are equal to each other, and, consequently, their opposite sides AB, AE are also equal (th. 4, B. I.).

But now, in the triangle CBE, the line AD, being drawn parallel to the side BE, cuts the two other sides CB, CE proportionally (th. 4), making CD to DB as is AC to AE, or to its equal AB. Q. E. D.

Case 2. The proposition is also applicable when an external angle of a triangle is bisected.



Let AC, one of the sides of the triangle ABC, be produced to E, and let the angle BCE be bisected by the straight line CD, cutting AB produced in D; then

AD : DB :: AC : CB.

Let BF be parallel to CD.

Then, because the line BC cuts the parallel lines CD, FB, it makes the angle CBF equal to the alternate angle BCD; and, therefore, also equal to the angle DCE, which is equal to BCD by supposition. Again, because the line EA cuts the two parallel lines CD, FB, it makes the angle DCE equal to the angle CFB, on the same side of the line.

Hence, in the triangle BCF, the angles BFC and FBC, being each equal to the angle DCE, are equal to each other; and, consequently, their opposite sides BC, CF are also equal.

Now, in the triangle ADC, the line BF, being drawn parallel to the side CD, cuts the two sides AD, AC proportionally, making

AD : AC :: DB : CF (th. 7, B. III.).

Or AD : DB :: AC : CF.

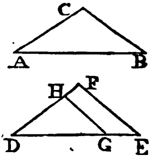
But BC is equal to CF; therefore,

AD : DB :: AC : CB.

THEOREM VI.

Equiangular triangles are similar, or have their like sides proportional.

Let ABC, DEF, be two equiangular triangles, having the angle A equal to the angle D, the angle B to the angle E, and, consequently, the angle C to the angle F; then will AB : AC :: DE : DF.



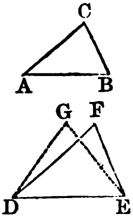
For, make DG = AB, and DH = AC, and join GH; then the two triangles ABC, DGH, having the two sides AB, AC equal to the two DG, DH, and the contained angles A and D also equal, are identical, or equal in all respects (th. 1, B. I.); namely, the angles B and C are equal to the angles G and H. But the angles B and C are equal to the angles E and F by the hypothesis; therefore, also, the angles G and H are equal to the angles E and F (ax. 1), and, consequently, the line GH is parallel to the side EF (Cor. 1, th. 15, B. I.).

Hence, then, in the triangle DEF, the line GH, being parallel to the side EF, divides the two other sides proportionally, making DG : DH :: DE : DF (Cor., th. 4). But DG and DH are equal to AB and AC; therefore, also, AB : AC :: DE : DF. Q. E. D.

THEOREM VII.

Triangles which have their sides proportional are also equiangular.

In the two triangles ABC, DEF, if AB : DE :: AC : DF :: BC : EF; the two triangles will have their corresponding angles equal.



For if the triangle ABC be not equiangular with the triangle DEF, suppose some other triangle, as DEG, to be equiangular with ABC. But this is impossible; for if the two triangles ABC, DEG were equiangular, their sides would be proportional (th. 6). So that, AB being to DE as AC to DG, and AB to DE as BC to EG, it follows that DG and EG, being fourth proportionals to the same three quantities, as well as the two DF, EF, the former, DG, EG, would be equal to the latter, DF, EF. Thus, then, the two triangles DEF, DEG, having their three sides equal, would be identical (th. 5, B. I.), which is absurd, since their angles are unequal.

THEOREM VIII.

Triangles which have an angle in the one equal to an angle in the other, and the sides about these angles proportional, are equiangular.

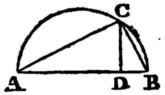
Let ABC, DEF, be two triangles, having the angle A = the angle D, and the sides AB, AC proportional to the sides DE, DF; then will the triangle ABC be equiangular with the triangle DEF.

For, make DG = AB, and DH = AC, and join GH.

Then the two triangles ABC, DGH, having two sides equal, and the contained angles A and D equal, are identical and equiangular (th. 1, B. I.), having the angles G and H equal to the angles B and C. But, since the sides DG, DH are proportional to the sides DE, DF, the line GH is parallel to EF (th. 4); hence the angles E and F are equal to the angles G and H (th. 15, B. I.), and, consequently, to their equals B and C. Q. E. D. [See fig. th. 6.]

THEOREM IX.

In a right-angled triangle, a perpendicular from the right angle is a mean proportional between the segments of the hypothenuse, and each of the sides about the right angle is a mean proportional between the hypothenuse and the adjacent segment.



Let ABC be a right-angled triangle, and CD a perpendicular from the right angle C to the hypothenuse AB; then will

CD be a mean proportional between AD and DB;
AC a mean proportional between AB and AD;
BC a mean proportional between AB and BD.

Or AD : CD :: CD : DB; and AB : BC :: BC : BD; and AB : AC :: AC : AD.

For the two triangles ABC, ADC, having the right angles at C and D equal, and the angle A common, have their third angles equal, and are equiangular (Cor. 1, th. 18, B. I.). In like manner, the two triangles ABC, BDC, having the right angles at C and D equal, and the angle B common, have their third angles equal, and are equiangular.

Hence, then, all the three triangles, ABC, ADC, BDC, being equiangular, will have their like sides proportional (th. 6),

viz., AD : CD :: CD : DB;
and AB : AC :: AC : AD;
and AB : BC :: BC : BD. Q. E. D.

Corol. 1. Because the angle in a semicircle is a right angle (th. 12, B. II.), it follows that, if from any point C in the periphery of the semicircle a perpendicular be drawn to the diameter AB, and the two chords CA, CB be drawn to the extremities of the diameter, then are AC, BC, CD the mean proportionals as in this theorem, or (by th. 12, B. III.), $CD^2 = AD.DB$; $AC^2 = AB.AD$; and $BC^2 = AB.BD$.

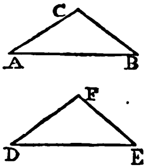
Corol. 2. Hence $AC^2 : BC^2 :: AD : BD$.

Corol. 3. Hence we have another demonstration of th. 34, B. I.

For, since $AC^2 = AB.AD$, and $BC^2 = AB.BD$;
By addition, $AC^2 + BC^2 = AB(AD + BD) = AB^2$.

THEOREM X.

Equiangular or similar triangles are to each other as the squares of their like sides.



Let ABC, DEF be two equiangular triangles, AB and DE being two like sides; then will the triangle ABC be to the triangle DEF as the square of AB is to the square of DE, or as AB^2 to DE^2.

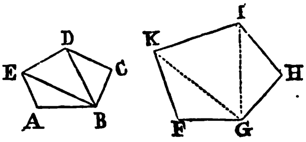
For, the triangles being similar, they have their like sides proportional (th. 6), and are to each other as the rectangles of the like pairs of their sides (Cor. 4, th. 3);

therefore AB : DE :: AC : DF (th. 6),
and AB : DE :: AB : DE of equality;
therefore $AB^2 : DE^2$ :: AB . AC : DE . DF (th. 10, B. III.).

But Δ ABC : Δ DEF :: AB.AC : DE.DF (Cor. 4, th. 3);
therefore Δ ABC : Δ DEF :: $AB^2 : DE^2$. Q. E. D.

THEOREM XI.

All similar figures are to each other as the squares of their like sides.



Let ABCDE, FGHIK be any two similar figures, the like sides being AB, FG, and BC, GH, and so on in the same order; then will the figure ABCDE be to the figure FGHIK as the square of AB to the square of FG, or as AB^2 to FG^2.

For, draw BE, BD, GK, GI, dividing the figures into an equal number of triangles by lines from two equal angles B and G.

The two figures being similar (by supposition), they are equiangular, and have their like sides proportional (def. 70.)

Then, since the angle A is = the angle F, and the sides AB, AE proportional to the sides FG, FK, the triangles ABE, FGK are equiangular (th. 8). In like manner, the two triangles BCD, GHI, having the angle C = the angle

H, and the sides BC, CD proportional to the sides GH, HI, are also equiangular. Also, if from the equal angles AED, FKI, there be taken the equal angles AEB, FKG, there will remain the equals BED, GKI; and if from the equal angles CDE, HIK be taken away the equals CDB, HIG, there will remain the equals BDE, GIK; so that the two triangles BDE, GIK, having two angles equal, are also equiangular. Hence each triangle of the one figure is equiangular with each corresponding triangle of the other.

But equiangular triangles are similar, and are proportional to the squares of their like sides (th. 10).

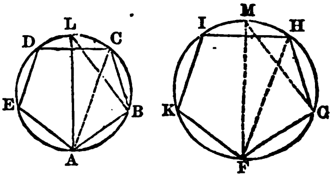
Therefore the Δ ABE : Δ FGK :: $AB^2 : FG^2$,
and Δ BCD : Δ GHI :: $BC^2 : GH^2$,
and Δ BDE : Δ GIK :: $DE^2 : IK^2$.

But as the two polygons are similar, their like sides are proportional, and, consequently, their squares also proportional; so that all the ratios AB^2 to FG^2, and BC^2 to GH^2, and DE^2 to IK^2, are equal among themselves, and, consequently, the corresponding triangles also, ABE to FGK, and BCD to GHI, and BDE to GIK, have all the same ratio, viz., that of AB^2 to FG^2; and hence all the antecedents, or the figure ABCDE, have to all the consequents, or the figure FGHIK, still the same ratio, viz., that of AB^2 to FG^2 (th. 7, B. III.). Q. E. D.

THEOREM XII.

Similar figures inscribed in circles have their like sides, and also their whole perimeters, in the same ratio as the diameters of the circles in which they are inscribed.

Let ABCDE, FGHIK be two similar figures inscribed in the circles whose diameters are AL and FM; then will each side AB, BC, &c., of the one figure be to the like side FG, GH, &c., of the other figure, or the whole perimeter AB+BC+, &c. of the one figure, to the whole perimeter FG+GH+, &c. of the other figure, as the diameter AL to the diameter FM.



For, draw the two corresponding diagonals, AC, FH, as also the lines BL, GM. Then, since the polygons are similar, they are equiangular, and their like sides have the same ratio (def. 70); therefore the two triangles ABC, FGH have the angle B=the angle G; and the sides AB, BC proportional to the two sides FG, GH; consequently, these two triangles are equiangular (th. 8), and have the angle ACB =FHG. But the angle ACB=ALB, standing on the same arc AB; and the angle FHG=FMG, standing on the same arc FG; therefore the angle ALB=FMG (ax. 1). And, since the angle ABL=FGM, being both right angles, because in a semicircle, therefore the two triangles ABL, FGM, having two angles equal, are equiangular; and, consequently, their like sides are proportional (th. 6); hence AB : FG :: the diameter AL : the diameter FM.

In like manner, each side BC, CD, &c., has to each side GH, HI, &c., the same ratio of AL to FM; and, consequently, the sums of them are still in the same ratio, viz., AB+BC+CD, &c. : FG+GH+HI, &c. :: the diam. AL : the diam. FM (th. 7, B. III.). Q. E. D.

THEOREM XIII.

Similar figures inscribed in circles are to each other as the squares of the diameters of those circles.

Let ABCDE, FGHIK be two similar figures inscribed in the circles whose diameters are AL and FM; then the surface of the polygon ABCDE will be to the surface of the polygon FGHIK as AL^2 to FM^2.

For the figures, being similar, are to each other as the squares of their like sides, AB^2 to FG^2 (th. 11). But, by the last theorem, the sides AB, FG are as the diameters AL, FM, and therefore the squares of the sides AB^2 to FG^2 as the squares of the diameters AL^2 to FM^2 (th. 9, B. III.). Consequently, the polygons ABCDE, FGHIK are also to each other as the squares of the diameters AL^2 to FM^2 (ax. 1). Q. E. D. [See fig. th. 12.]

THEOREM XIV.

The circumferences of all circles are to each other as their diameters.

Let D, d denote the diameters of two circles, and C, c their circumferences; then will $D : d :: C : c$, or $D : C :: d : c$.

For (by th. 12) similar polygons inscribed in circles have their perimeters in the same ratio as the diameters of those circles.

Now, as this property belongs to all polygons, whatever the number of the sides may be, conceive the number of the sides to be indefinitely great, and the length of each infinitely small, till they coincide with the circumference of the circle, and be equal to it, indefinitely near; then the perimeter of the polygon of an indefinite number of sides is the same thing as the circumference of the circle. Hence it appears that the circumferences of the circles, being the same as the perimeters of such polygons, are to each other in the same ratio as the diameters of the circles. Q. E. D.

THEOREM XV.

The areas or spaces of circles are to each other as the squares of their diameters or of their radii.

Let A, a denote the areas or spaces of two circles, and D, d their diameters; then $A : a :: D^2 : d^2$.

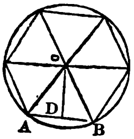
For (by th. 13) similar polygons inscribed in circles are to each other as the squares of the diameters of the circles.

Hence, conceiving the number of the sides of the polygons to be increased more and more, or the length of the sides to become less and less, the polygon approaches nearer and nearer to the circle, till at length, by an infinite approach, they coincide, and become in effect equal; and then it follows that the spaces of the circles, which are the same as of the polygons, will be to each other as the squares of the diameters of the circles. Q. E. D.

Corol. The spaces of circles are also to each other as the squares of the circumferences, since the circumferences are in the same ratio as the diameters (by th. 14).

THEOREM XVI.

The area of any circle is equal to the rectangle of half its circumference and half its diameter.



Conceive a regular polygon to be inscribed in a circle, and radii drawn to all the angular points, dividing it into as many equal triangles as the polygon has sides, one of which is ABO, of which the altitude is the perpendicular OD from the centre to the base AB.

Then the triangle ABO, being equal to a rectangle of half the base and equal altitude (Cor. 2, th. 26, B. I.), is equal to the rectangle of the half base AD and the altitude OD; consequently, the whole polygon, or all the triangles added together which compose it, is equal to the rectangle of the common altitude OD, and the halves of all the sides, or the half perimeter of the polygon.

Now conceive the number of sides of the polygon to be indefinitely increased; then will its perimeter coincide with the circumference of the circle, and, consequently, the altitude OD will become equal to the radius, and the whole polygon equal to the circle. Consequently, the space of the circle, or of the polygon in that state, is equal to the rectangle of the radius and half the circumference. Q. E. D.

THEOREM XVII.

The area of a circle is equal to the square of the radius multiplied by the ratio of the diameter to the circumference.

Let the circumference of the circle whose diameter is unity be denoted by π; then, since the diameters of circles are to each other as their circumferences (Prop. 14), π will denote the ratio of any circumference to its diameters. We shall then have (r being the radius),

$$1 : \pi :: 2r : \text{circumference};$$

$$\therefore \text{circumference} = \pi \times 2r.$$

Multiplying both members by $\frac{1}{2}r$, we have

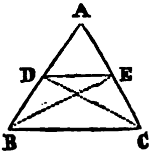
$$\tfrac{1}{2}r \times \text{circumference} = \pi \,.\, r^2;$$

or (th. 16) $$\text{area} = \pi \,.\, r^2.$$

The numerical value of π is 3.14159.

THEOREM XVIII.

If a straight line cut two sides of a triangle proportionally, this line will be parallel to the third side. (Converse of th. 4.)



Let the line DE cut the sides AB, AC, so that AD : DB :: AE : EC; then will the line DE be parallel to the side BC.

Because the two triangles ADE, EDB, have their bases in the same straight line AB, and their vertices at E, they have the same altitude, and are to each other as their bases (th. 1).

Or $\qquad ADE : EDB :: AD : DB.$

And, for a like reason,

$$ADE : EDC :: AE : EC.$$

But $\qquad AD : DB :: AE : EC;$

$$\therefore ADE : EDB :: ADE : EDC.$$

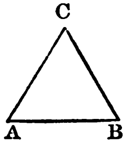
Hence, EDB=EDC; and, having the same base DE, they must have the same altitude;

$\therefore$ DE is parallel to BC. Q. E. D.

PROBLEMS.

PROBLEM I.

To make an equilateral triangle on a given line AB.

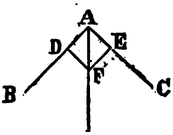


From the centres A and B, with the distance AB, describe arcs intersecting in C. Draw AC, BC, and ABC will be the equilateral triangle.

For the equal radii AC, BC are each of them equal to AB.

PROBLEM II.

To bisect a given angle BAC.



From the centre A, with any radius, describe an arc cutting off the equal lines AD, AE; and from the two centres, D, E, with the same radius, describe arcs intersecting in F; then draw AF, which will bisect the angle A as required.

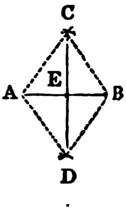
Join DF, EF. Then the two triangles ADF, AEF, having the two sides AD, DF equal to the two AE, EF (being equal radii), and the side AF common, they are mutually equilateral; consequently, they are also mutually equiangular (th. 5), and have the angle BAF equal to the angle CAF.

Scholium. In the same manner is an arc of a circle bisected.

PROBLEM III.

To bisect a given line AB.

From the two centres A and B, with any equal radii, describe arcs of circles intersecting each other in C and D,



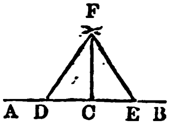
and draw the line CD, which will bisect the given line AB in the point E.

Draw the radii AC, BC, AD, BD. Then, because all these four radii are equal, and the side CD common, the two triangles ACD, BCD are mutually equilateral; consequently, they are also mutually equiangular (th. 5, B. I.), and have the angle ACE equal to the angle BCE.

Hence the two triangles ACE, BCE, having the two sides AC, CE equal to the two sides BC, CE, and their contained angles equal, are identical (th. 1), and therefore have the side AE equal to EB.

PROBLEM IV.

At a given point C, *in a line* AB, *to erect a perpendicular.*

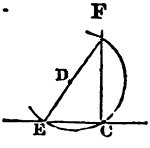


From the given point C, with any radius, cut off any equal parts CD, CE of the given line; and from the two centres D and E, with any one radius, describe arcs intersecting in F; then join CF, which will be perpendicular as required.

Draw the two equal radii DF, EF. Then the two triangles CDF, CEF, having the two sides CD, DF, equal to the two CE, EF, and CF common, are mutually equilateral; consequently, they are also mutually equiangular (th. 5, B. I.), and have the two adjacent angles at C equal to each other; therefore the line CF is perpendicular to AB (def. 11).

OTHERWISE.

When the point C *is near the end of the line.*

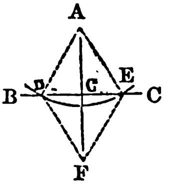


From any point D, assumed above the line, as a centre, through the given point C describe a circle, cutting the given line at E; and through E and the centre D, draw the diameter EDF; then join CF, which will be the perpendicular required.

For the angle at C, being an angle in a semicircle, is a right angle, and therefore the line CF is a perpendicular (by def. 15).

PROBLEM V.

From a given point A, *to let fall a perpendicular on a given line* BC.

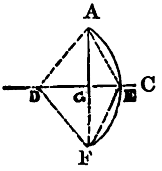


From the given point A as a centre, with any convenient radius, describe an arc, cutting the given line at the two points D and E; and from the two centres D, E, with any radius, describe two arcs, intersecting at F; then draw AGF, which will be perpendicular to BC, as required.

Draw the equal radii AD, AE, and DF, EF. Then the two triangles ADF, AEF, having the two sides AD, DF, equal to the two AE, EF, and AF common, are mutually equilateral; consequently, they are also mutually equiangular (th. 5, B. I.), and have the angle DAG equal the angle EAG. Hence, then, the two triangles ADG, AEG, having the two sides AD, AG, equal to the two AE, AG, and their included angles equal, are therefore equiangular (th. 1, B. I.), and have the angles at G equal; consequently, AG is perpendicular to BC (def. 11).

OTHERWISE.

When the point is nearly opposite the end of the line.



From any point D, in the given line BC, as a centre, describe the arc of a circle through the given point A, cutting BC in E; and from the centre E, with the radius EA, describe another arc, cutting the former in F; then draw AGF, which will be perpendicular to BC as required.

Draw the equal radii DA, DF, and EA, EF. Then the two triangles DAE, DFE, will be mutually equilateral; consequently, they are also mutually equiangular (th. 5, B. I.), and have the angles at D equal. Hence the two triangles DAG, DFG, having the two sides DA, DG equal to the two DF, DG, and the included angles at D equal, have also the angles at G equal (th. 1, B. I.); consequently, those angles at G are right angles, and the line AG is perpendicular to DG.

PROBLEM VI.

To make a triangle with three given lines, AB, AC, BC.

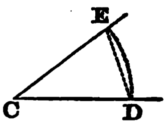
With the centre A, and distance AC, describe an arc. With the centre B, and distance BC, describe another arc, cutting the former in C. Draw AC, BC, and ABC will be the triangle required.

For the radii, or sides of the triangle, AC, BC, are equal to the given lines AC, BC, by construction.

Note.—If any two of the lines are not together greater than the third, the construction is impossible.

PROBLEM VII.

At a given point A, *in a line* AB, *to make an angle equal to a given angle* C.

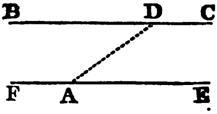


From the centres A and C, with any one radius, describe the arcs DE, FG. Then, with radius DE and centre F, describe an arc, cutting FG in G. Through G draw the line AG, and it will form the angle required.

Conceive the equal lines or radii, DE, FG, to be drawn. Then the two triangles CDE, AFG, being mutually equilateral, are mutually equiangular (th. 5, B. I.), and have the angle at A equal to the angle at C.

PROBLEM VIII.

Through a given point A, *to draw a line parallel to a given line* BC.

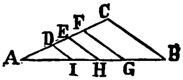


From the given point A draw a line AD to any point in the given line BC. Then draw the line EAF, making the angle at A equal to the angle at D (by Prob. 5); so shall EF be parallel to BC, as required.

For the angle D being equal to the alternate angle A, the lines BC, EF are parallel, by th. 13, B. I.

PROBLEM IX.

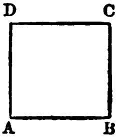
To divide a line AB *into any proposed number of equal parts.*



Draw any other line AC, forming any angle with the given line AB, on which set off as many equal parts AD, DE, EF, FC, as the line AB is to be divided into. Join BC; parallel to which draw the other lines FG, EH, DI: then these will divide AB in the manner required. For those parallel lines divide both the sides AB, AC proportionally, by th. 4, B. IV.

PROBLEM X.

To make a square on a given line AB.

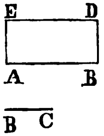


Raise AD, BC, each perpendicular and equal to AB, and join CD; so shall ABCD be the square sought.

For all the three sides AB, AD, BC are equal by the construction, and DC is equal and parallel to AB (by th. 24, B. I.); so that all the four sides are equal, and the opposite ones are parallel. Again, the angle A or B of the parallelogram being a right angle, the angles are all right ones (Cor. 1, th. 22, B. I.). Hence, then, the figure, having all its sides equal and all its angles right, is a square (def. 34).

PROBLEM XI.

To make a rectangle or a parallelogram of a given length and breadth, AB, BC.



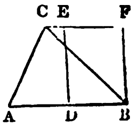
Erect AE, BD perpendicular to AB, and each equal to BC; then join DE, and it is done.

The demonstration is the same as the last problem.

And in the same manner is described any oblique parallelogram, only drawing AE and BD to make the given oblique angle with AB instead of perpendicular to it.

PROBLEM XII.

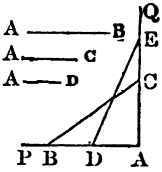
To make a rectangle equal to a given triangle ABC.



Bisect the base AB in D; then raise DE and BF perpendicular to AB, and meeting CF parallel to AB at E and F; so shall DF be the rectangle equal to the given triangle ABC (by Cor. 2, th. 26, B. I.).

PROBLEM XIII.

To make a square equal to the sum of two or more given squares.

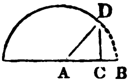


Let AB and AC be the sides of two given squares. Draw two indefinite lines AP, AQ, at right angles to each other, in which place the sides AB, AC of the given squares; join BC; then a square described on BC will be equal to the sum of the two squares described on AB and AC (th. 34, B. I.).

In the same manner, a square may be made equal to the sum of three or more given squares. For, if AB, AC, AD be taken as the sides of the given squares, then, making AE =BC, AD=AD, and drawing DE, it is evident that the square on DE will be equal to the sum of the squares on AB, AC, AD. And so on for more squares.

PROBLEM XIV.

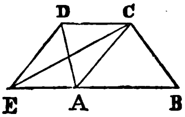
To make a square equal to the difference of two given squares.



Let AB and AC, taken in the same straight line, be equal to t[illegible] ides of two given squares. Fr[illegible] the centr[illegible] h the distance AB, de- s[illegible] a circl[illegible] ake CD perpendicular [illegible] mee[illegible] ence in D; so shall a square [illegible] D^2-AC^2, or AB^2-AC^2, as [illegible]

PROBLEM XV.

To make a triangle equal to a given quadrilateral ABCD.

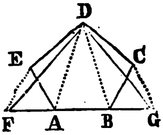


Draw the diagonal AC, and parallel to it DE, meeting BA produced at E, and join CE; then will the triangle CEB be equal to the given quadrilateral ABCD.

For the two triangles ACE, ACD, being on the same base AC, and between the same parallels AC, DE, are equal (th. 25, B. I.); therefore, if ABC be added to each, it will make BCE equal to ABCD (ax. 2).

PROBLEM XVI.

To make a triangle equal to a given pentagon ABCDE.

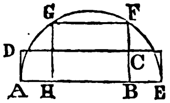


Draw DA and DB, and also EF, CG, parallel to them, meeting AB produced at F and G; then draw DF and DG; so shall the triangle DFG be equal to the given pentagon ABCDE.

For the triangle DFA=DEA, and the triangle DGB=DCB (th. 25); therefore, by adding DAB to the equals, the sums are equal (ax. 2); that is, DAB + DAF + DBG = DAB+DAE+DBC, or the triangle DFG=to the pentagon ABCDE.

PROBLEM XVII.

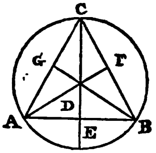
To make a square equal to a given rectangle ABCD.



Produce one side AB, till BE be equal to the other side BC. On AE as a diameter describe a circle meeting BC produced at F; then will BF be the side of the square BFGH, equal to the given rectangle BD, as required, as appears by Cor., th. 9, B. IV., and th. 12, B. III.

PROBLEM XVIII.

To describe a circle about a given triangle ABC.

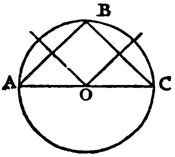


Bisect any two sides with two of the perpendiculars, DE, DF, DG, and D will be the centre.

For, join DA, DB, DC. Then the two right-angled triangles DAE, DBE, have the two sides DE, EA equal to the two DE, EB, and the included angles at E equal; these two triangles are therefore identical (th. 1, B. I.), and have the side DA equal to DB. In like manner it is shown that DC is also equal to DA or DB. So that all the three, DA, DB, DC, being equal, they are radii of a circle passing through A, B, and C.

Note.—The problem is the same in effect when it is required

To describe the circumference of a circle through three given points A, B, C.



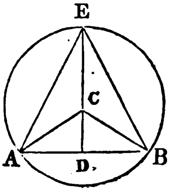
Then, from the point B draw chords BA, BC, to the two other points, and bisect these chords perpendicularly by lines meeting in O, which will be the centre. Again, from the centre O, at the distance of any one of the points, as OA, describe a circle, and it will pass through the two other points, B, C, as required. The demonstration is evidently as above.

PROBLEM XIX.

An isosceles triangle ABC *being given, to describe another on the same base* AB, *whose vertical angle shall be only half the vertical angle* C.

From C as a centre, with the distance CA, describe the circle ABE. Bisect AB in D, join DC, and produce to the circumference E; join EA and EB, and ABE shall be the isosceles triangle required.

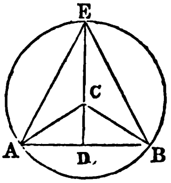
For, since in the triangles EDA, EDB, AD is equal to



DB, and DE common to both, and the right angle EDA equal to the right angle EDB, the side EA must be equal to the side EB; the triangle AEB is therefore isosceles, and the angle ACB at the centre must be double of the angle AEB at the circumference, for they both stand on the same segment AB.

PROBLEM XX.

Given an isosceles triangle AEB, *to erect another on the same base* AB, *which shall have double the vertical angle* E.

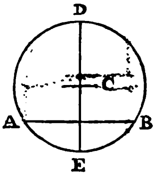


Describe a circle about the triangle AEB; find its centre C, and join CA, AB, and ACB is the triangle required.

The angle C at the centre is double of the angle E at the circumference, and the triangle ACB is isosceles, for the sides CA, CB, being radii of the same circle, are equal.

PROBLEM XXI.

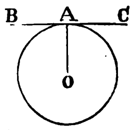
To find the centre of a given circle.



Draw any chord AB, and bisect it perpendicularly with the line ED; this (Cor., th. 1, B. II.) will be a diameter. Therefore bisect ED in C, which will be the centre, as required.

PROBLEM XXII.

To draw a tangent to a circle through a given point A.

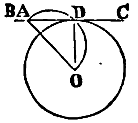


1. When the given point A is in the circumference of the circle, join A and the centre O, perpendicular to which draw BAC, and it will be the tangent, by th. 6, B. II.

2. When the given point A is out of the circle, draw AO to the centre, on which, as a diameter, de-

scribe a semicircle, cutting the given circumference in D, through which draw BADC, which will be the tangent as required.

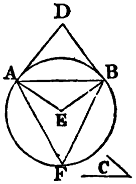
For, join DO. Then the angle ADO, in a semicircle, is a right angle, and, consequently, AD is perpendicular to the radius DO, or is a tangent to the circle (th. 6, B. II.).



PROBLEM XXIII.

On a given line AB *to describe a segment of a circle, to contain a given angle* C.

At the ends of the given line make angles DAB, DBA, each equal to the given angle C. Then draw AE, BE, perpendicular to AD, BD; and with the centre E, and radius EA, or EB, describe a circle; so shall AFB be the segment required, as any angle F made in it will be equal to the given angle C.

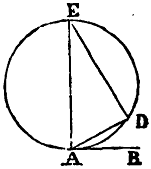


For, the two lines AD, BD, being perpendicular to the radii EA, EB (by construction), are tangents to the circle (th. 6, B. II.); and the angle A or B, which is equal to the given angle C by construction, is equal to the angle F in the alternate segment AEB (th. 13, B. II.).

PROBLEM XXIV.

To cut off a segment from a circle that shall contain a given angle C.

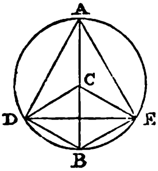
Draw any tangent AB to the given circle, and a chord AD to make the angle DAB equal to the given angle C; then DEA will be the segment required, any angle E made in it being equal to the given angle C.



PROBLEM XXV.

To inscribe an equilateral triangle in a given circle.

Through the centre C draw any diameter AB. From the point B as a centre, with the radius BC of the given

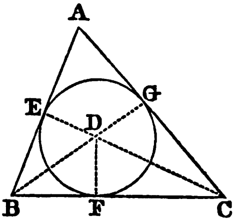


circle, describe an arc DCE. Join AD, AE, DE, and ADE is the equilateral sought.

Join DB, DC, EB, EC. Then DCB is an equilateral triangle, having each side equal to the radius of the given circle. In like manner, BCE is an equilateral triangle. But the angle ADE is equal to the angle ABE or CBE, standing on the same arc AE; also the angle AED is equal to the angle CBD, on the same are AD; hence the triangle DAE has two of its angles, ADE, AED, equal to the angles of an equilateral triangle, and therefore the third angle at A is also equal to the same, so that the triangle is equiangular, and therefore equilateral.

PROBLEM XXVI.

To inscribe a circle in a given triangle ABD.



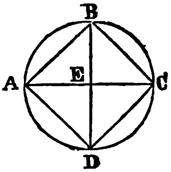
Bisect any two angles C and B with the two lines CD, BD. From the intersection D, which will be the centre of the circle, draw the perpendiculars DE, DF, DG, and they will be the radii of the circle required.

For, since the angle DCF is equal to the angle DCG, and the angles at F, G, right angles (by construction), the two triangles CDF, CDG are equiangular; and, having also the side CD common, they are identical, and have the sides DE, DG equal (th. 2, B. I.). In like manner it is shown that DF is equal to DE or DG.

Therefore, if with the centre D, and distance DE, a circle be described, it will pass through all the three points E, F, G, in which points also it will touch the three sides of the triangle (th. 6, B. II.), because the radii DE, DF, DG, are perpendicular to them.

PROBLEM XXVII.

To inscribe a square in a given circle.

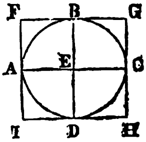


Draw two diameters AC, BD, crossing at right angles in the centre E. Then join the four extremities A, B, C, D, with right lines, and these will form the inscribed square ABCD.

For the four right-angled triangles AEB, BEC, CED, DEA, are identical, because they have the sides EA, EB, EC, ED all equal, being radii of the circle, and the four included angles at E all equal, being right angles, by the construction. Therefore all their third sides AB, BC, CD, DA, are equal to one another, and the figure ABCD is equilateral. Also, all its four angles, A, B, C, D, are right ones, being angles in a semicircle. Consequently the figure is a square.

PROBLEM XXVIII.

To describe a square about a given circle.



Draw two diameters AC, BD, crossing at right angles in the centre E. Then through their four extremities draw FG, IH, parallel to AC, and FI, GH, parallel to BD, and they will form the square FGHI.

For, the opposite sides of parallelograms being equal, FG and IH are each equal to the diameter AC, FI and GH each equal to the diameter BD, so that the figure is equilateral. Again, because the opposite angles of parallelograms are equal, all the four angles F, G, H, I are right angles, being equal to the opposite angles at E. So that the figure FGHI, having its sides equal, and its angles right ones, is a square, and its sides touch the circle at the four points A, B, C, D, being perpendicular to the radii drawn to those points.

PROBLEM XXIX.

To inscribe a circle in a given square.

Bisect the two sides FI, FG, in the points A and B (last fig.). Then, through these two points draw AC parallel to FG or IH, and BD parallel to FI or GH. Then the point of intersection E will be the centre, and the four lines EA, EB, EC, ED, radii of the inscribed circle.

For, because the four parallelograms EF, EG, EH, EI, have their opposite sides and angles equal, therefore all the four lines EA, EB, EC, ED are equal, being each equal to half a side of the square. So that a circle described from the centre E, with the distance EA, will pass through all the points A, B, C, D, and will be inscribed in the square, or will touch its four sides in those points, because the angles there are right ones.

PROBLEM XXX.

To describe a circle about a given square.

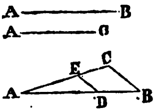
(See figure, Problem xxvii.)

Draw the diagonals AC, BD, and their intersection E will be the centre.

As the diagonals of a square bisect each other (th. 40), then will EA, EB, EC, ED be all equal, and, consequently, these are radii of a circle passing through the four points A, B, C, D.

PROBLEM XXXI.

To find a third proportional to two given lines, AB, AC.

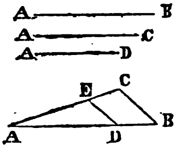


Place the two given lines AB, AC (or two lines equal to them) to form any angle at A; and in AB set off AD=AC. Join BC, and draw DE parallel to it; so will AE, on the line AC, be the third proportional sought.

For, since DE is parallel to BC, the two lines AB, AC are cut proportionally by DE (th. 4, B. IV.); hence AB : AC :: AD (=AC) : AE, and AE is, therefore, the third proportional required.

PROBLEM XXXII.

To find a fourth proportional to three given lines, AB, AC, AD.

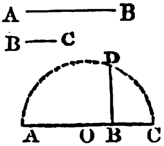


Place two of the given lines, AB, AC, or their equals, to make any angle at A; and on AB set off, or place the other line AD, or its equal. Join BC, and parallel to it draw DE; so shall AE be the fourth proportional as required.

For, because of the parallels BC, DE, the two sides AB, AC, are cut proportionally (th. 4, B. IV.); so that AB : AC :: AD : AE.

PROBLEM XXXIII.

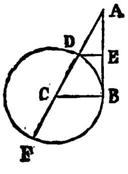
To find a mean proportional between two lines, AB, BC.



Place AB, BC, joined in one straight line AC; on which, as a diameter, describe the semicircle ADC; to meet which erect the perpendicular BD, and it will be the mean proportional sought between AB and BC, by Cor., th. 9, B. IV.

PROBLEM XXXIV.

To divide a given line in extreme and mean ratio.



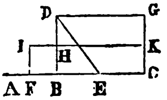
Let AB be the given line to be divided in extreme and mean ratio, that is, so that the whole line may be to the greater part as the greater is to the less part.

Draw BC perpendicular to AB, and equal to half AB. Join AC; and with centre C and distance CB, describe the circle BD; then, with centre A and distance AD, describe the arc DE; so shall AB be divided in E in extreme and mean ratio, or so that AB : AE :: AE : EB.

Produce AC to the circumference at F. Then, ADF being a secant, and AB a tangent, because B is a right angle; therefore the rectangle AF . AD is equal to AB^2 (Cor. 1, th. 21); consequently the means and extremes of these are proportional (th. 12, B. III.), viz., AB : AF or AD+DF :: AD : AB. But AE is equal to AD by construction, and AB=2BC=DF; therefore AB : AE+AB :: AE : AB or AE+EB; and by division, AB : AE :: AE : EB.

PROBLEM XXXV.

To cut a given line AB *in a point* F, *so that the square of the one part* BF *may be equal to the rectangle of the whole line* AB *and the other part* AF.

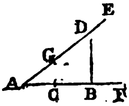


Produce AB till BC be equal to it; erect the perpendicular BD equal to AB or BC; bisect BC in E; join ED, and make EF equal to it; the square of the segment BF is equivalent to the rectangle contained by the whole BA and its remaining segment AF. The line AB is then said to be divided by medial section at the point F.

For on BC construct the square BG; make BH equal to BF, and draw IHK and FI parallel to AC and BD. Since AB is equal to BD, and BF to BH, the remainder AF is equal to HD; and it is further evident that FH is a square, and IC and DK are rectangles. But BC being bisected in E and produced to F, the rectangle under BF, FC, or the rectangle IC, together with the square of BE, is equivalent to the square of EF or DE. But the square of DE is equivalent to the squares of DB and BE; whence the rectangle IC, with the square of BE, is equivalent to the squares of DB and BE; or, omitting the common square of BE, the rectangle IC is=to the square of DB. Take away from both the rectangle BK, and there remains the square BI or the square of BF=to the rectangle HG, or the rectangle contained by BA and AF.

Cor. Hence also the construction of another problem of the same nature, in which it is required to produce a straight line AB, such that the rectangle contained by the whole line thus produced and the part produced shall be equivalent to the square of the line AB itself.

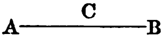
Bisect AB in C; draw the perpendicular BD=BC; join AD, and continue it until DE=DB or BC, and on AB produced take AF=AE; the line AF is the required exten-



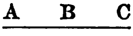
sion of AB. For, make DG=DB or BC; and because the rectangle EA, AG, together with the square of DG or DB, is equivalent to the square of DA or to the squares of AB and DB, the rectangle EA, AG, or FA, AC, is equivalent to the square of AB.

PROBLEM XXXVI.

Given either one of the sides AB, *or the base* a b, *to construct an isosceles triangle, so that each of the angles at the base may be double of its vertical angle.*



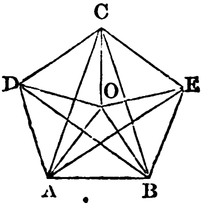
First, let one of the sides AB be given. By the last problem divide it into two parts, AC, CB, such that $CB^2=AB\times AC$. Construct the triangle, having the base=CB, and each of the two sides=AB.



Next, if the base AB be given, by the second case of the foregoing proposition, produce AB to C, so that $AC\times CB=AB^2$; then will AB be the base, and AC the length of each of the two sides.

PROBLEM XXXVII.

To describe a regular pentagon on a given line AB.



On AB erect the isosceles triangle ACB, having each of the angles at the base double of its vertical angle; on AB again construct another isosceles triangle, whose vertical angle AOB is double of ACB, and about the vertex O place the isosceles triangles AOD, DOC, COE, and EOB; these triangles, with AOB, will compose a regular pentagon.

For the angle AOB, being the double of ACB, which is the fifth part of two right angles, must be equal to the fifth part of four right angles; and, consequently, five angles, each of them equal to AOB, will adapt themselves about the point O. But the bases of those central triangles, and which form the sides of the pentagon, are all equal; and the angles at their bases being likewise equal, they are equal in the collective pairs which constitute the internal angles of the figure. It is therefore a regular pentagon.

PROBLEM XXXVIII.

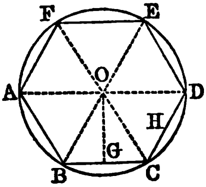
To describe a hexagon upon a given line AB.

From A and B as centres, with AB as radius, describe arcs intersecting in O (fig. to the next problem). From O as a centre, with the same radius, describe a circle ABCDEF. Within this circle set off from B the chords BC, CD, DE, EF, FA, in succession, each equal to AB; they will, together with AB, form the hexagon required.

The demonstration is analogous to that of the following problem.

PROBLEM XXXIX.

To inscribe a regular hexagon in a circle.



Apply the radius AO of the given circle as a chord, AB, BC, CD, &c., quite round the circumference, and it will complete the regular hexagon ABCDEF.

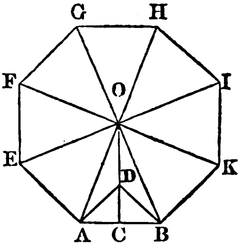
For, draw the radii AO, BO, CO, DO, EO, FO, completing six equal triangles, of which any one, as ABO, being equilateral (by constr.), its three angles are all equal (Cor. 2, th. 3, B. I.), and any one of them, as AOB, is one third of the whole, or of two right angles (th. 17, B. I.), or one sixth of four right angles. But the whole circumference is the measure of four right angles (Cor. 4, th. 6, B. I.). Therefore the arc AB is one sixth of the circumference of the circle, and, consequently, its chord AB one side of an equilateral hexagon inscribed in the circle. And the same of the other chords.

Cor. The side of a regular hexagon is equal to the radius of the circumscribing circle, or to the chord of one sixth part of the circumference.

PROBLEM XL.

On a given line AB, *to construct a regular octagon.*

Bisect AB by the perpendicular CD, which make=CA

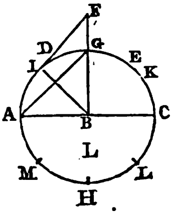


or CB; join DA and DB; produce CD, making DO=DA or DB; draw AO and BO, thus forming an angle equal to the half of ADB, and about the vertex O repeat the equal triangles AOB, AOE, EOF, FOG, GOH, HOI, IOK, and KOB, to compose the octagon.

For the distances AD, BD are evidently equal; and because CA, CD, and CB are all equal, the angle ADB is contained in a semicircle, and is, therefore, a right angle. Consequently, AOB is equal to the half of a right angle, and eight such angles will adapt themselves about the point O. Whence the figure BAEFGHIK, having eight equal sides and equal angles, is a regular octagon.

PROBLEM XLI.

To divide the circumference of a given circle successively into 4, 8, 12, *and* 24 *equal parts.*



1. Insert the radius AB three times from A to D, E, and C; from the extremities of the diameter AC, and with a distance equal to the double chord AE, describe arcs intersecting in the point F; and from A, with the distance BF, cut the circumference on opposite sides at G and H; AG, GC, CH, and HA are quadrants.

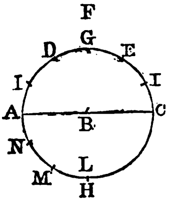
2. From the point F, with the radius AB cut the circle in I and K, and from A and C inflect the chord AI from L and M; the circumference is divided into eight equal portions by the points A, I, G, K, C, M, H, and L.

3. The arc DG, on being repeated, will form twelve equal sections of the circumference.

4. The arc ID is the twenty-fourth part of the circumference.

PROBLEM XLII.

To divide the circumference of a given circle successively into 5, 10, and 20 equal parts.

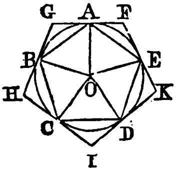


Mark out the semicircumference ADEC by the triple insertion of the radius; from A and C, with the double chord AE, describe arcs intersecting in F; from A, with the distance BF, cut the circle in G and H; inflect the chords GI and GI equal to the radius AB; and from the points I and I, with distance BF or AG, describe arcs intersecting in L.

For BL is the greater segment of the radius BH divided by a medial section; wherefore AL is equal to the side of the inscribed pentagon, and BL to that of the decagon inscribed in the given circle. Hence AL may be inflected five times in the circumference, and BL ten times; and, consequently, the arc MK, or the excess of the fourth above the fifth, is equal to the twentieth part of the whole circumference.

PROBLEM XLIII.

To describe a regular pentagon, hexagon, or octagon about a circle.



In the given circle inscribe a regular polygon of the same name or number of sides, as ABCDE, by one of the foregoing problems. Then to all its angular points draw tangents (by prob. 22), and these will form the circumscribing polygon required.

For all the chords or sides of the inscribed figure, AB, BC, &c., being equal, and all the radii OA, OB, &c., being equal, all the vertical angles about the point O are equal. But the angles OEF, OAF, OAG, OBG, made by the tangents and radii, are right angles; therefore OEF+OAF=two right angles, and OAG+OBG =two right angles; consequently, also, AOE+AFE=two right angles, and AOB+AGB=two right angles (Cor. 2, th. 18, B. I.). Hence, then, the angles AOE+AFE being =AOB+AGB, of which AOB is =AOE, consequently the remaining angles F and G are also equal. In the same manner it is shown that all the angles F, G, H, I, K are equal.

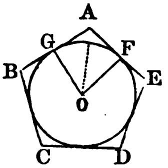
Again, the tangents from the same point FE, FA are equal, as also the tangents AG, GB (Cor. 2, th. 21, B. II.); and the angles F and G of the isosceles triangles AFE, AGB, are equal, as well as their opposite sides AE, AB; consequently, those two triangles are identical (th. 1), and have their other sides EF, FA, AG, GB, all equal, and FG equal to the double of any one of them. In like manner it is shown that all the other sides GH, HI, IK, KF, are equal to FG, or double of the tangents GB, BH, &c.

Hence, then, the circumscribed figure is both equilateral and equiangular, which was to be shown.

Cor. The inscribed circle touches the middle of the sides of the polygon.

PROBLEM XLIV.

To inscribe a circle in a regular polygon.

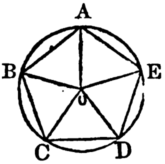


Bisect any two sides of the polygon by the perpendiculars GO, FO, and their intersection O will be the centre of the inscribed circle, and OG or OF will be the radius.

For the perpendiculars to the tangents AF, AG pass through the centre (Cor., th. 7, B. II.), and the inscribed circle touches the middle points F, G by the last corollary. Also, the two sides AG, AO, of the right-angled triangle AOG, being equal to the two sides AF, AO of the right-angled triangle AOF, the third sides OF, OG will also be equal. Therefore the circle described with the centre O and radius OG will pass through F, and will touch the sides in the points G and F. And the same for all the other sides of the figure.

PROBLEM XLV.

To describe a circle about a regular polygon.

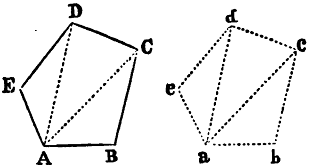


Bisect any two of the angles C and D with the lines CO, DO; then their intersection O will be the centre of the circumscribing circle, and OC or OD will be the radius.

For, draw OB, OA, OE, &c., to the angular points of the given polygon. Then the triangle OCD is isosceles, having the angles at C and D equal, being the halves of the equal angles of the polygon BCD, CDE; therefore their opposite sides CO, DO are equal (th. 4, B. I.). But the two triangles OCD, OCB, having the two sides OC, CD equal to the two OC, CB, and the included angles OCD, OCB also equal, will be identical (th. 1, B. I.), and have their third sides BO, OD equal. In like manner it is shown that all the lines OA, OB, OC, OD, OE, are equal. Consequently, a circle described with the centre O and radius OA will pass through all the other angular points B, C, D, &c., and will circumscribe the polygon.

PROBLEM XLVI.

On a given line, to construct a rectilinear figure similar to a given rectilinear figure.

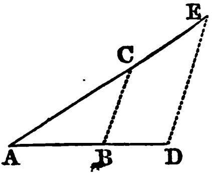


Let *abcde* be the given rectilinear figure, and AB the side of the proposed similar figure that is similarly posited with *ab*.

Place AB in the prolongation of *ab*, or parallel to it. Draw AC, AD, AE, &c., parallel to *ac*, *ad*, *ae* respectively. Draw BC parallel to *bc*, meeting AC in C; CD parallel to *cd*, and meeting AD in D; DE parallel to *de*, and meeting AE in E, and so on till the figure is completed. Then ABCDE will be similar to *abcde*, from the nature of parallel lines and similar figures (th. 11, B. IV.).

PROBLEM XLVII.

Through a given point within an angle, to draw a straight line such that the parts included between the point and the sides of the angle may be equal.



Let B be the given point, and AED the angle.

It is required to draw a line AD, through B, so that AB may be equal to BD.

Through the point B draw BC parallel to DE, and take CA=CE.

Through the points A and B draw the line ABD, which will be the line required.

For BC is parallel to DE by construction.

Therefore AC : CE :: AB : BD (th. 4, B. IV.).

But AC is equal to CE by construction;

∴ AB is equal to BD.

GEOMETRY OF PLANES.

BOOK V.

DEFINITIONS.

1. A **PLANE** is a surface in which, if any two points be taken, the straight line which joins these points will be wholly in that surface.

2. A straight line is said to be perpendicular to a plane when it is perpendicular to all the straight lines in the plane which pass through the point in which it meets the plane.

This point is called the foot of the perpendicular.

3. The inclination of a straight line to a plane is the acute angle contained by the straight line and another straight line drawn from the point in which the first meets the plane to the point in which a perpendicular to the plane, drawn from any point in the first line, meets the plane.

4. A straight line is said to be parallel to a plane when it can not meet the plane, to whatever distance both be produced.

5. It will be proved in Prop. 2, that the common intersection of two planes is a straight line; this being premised,

The angle contained by two planes which cut one another is measured by the angle contained by two straight lines drawn from any point in the common intersection of the planes perpendicular to it, one in each of the planes.

This angle may be acute, right, or obtuse.

If it be a right angle, the planes are said to be perpendicular to each other.

6. Two planes are parallel to each other when they can not meet, to whatever distance both be produced.

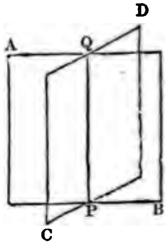
PROPOSITION I.

A straight line can not be partly in a plane and partly out of it.

For, by Def. 1, when a straight line has two points common to a plane, it lies wholly in that plane.

PROPOSITION II.

If two planes cut each other, their common intersection is a straight line.



Let the two planes AB, CD cut one another, and let P, Q be two points in their common section.

Join P, Q;

Then, since the points P, Q are in the same plane AB, the straight line PQ which joins them must lie wholly in that plane.

For a similar reason, PQ must lie wholly in the plane CD.

∴ The straight line PQ is common to the two planes, and is ∴ their common intersection.

PROPOSITION III.

Any number of planes may be drawn through the same straight line.

For let a plane, drawn through a straight line, be conceived to revolve round the straight line as an axis. Then the different positions assumed by the revolving plane will be those of different planes drawn through the straight line.

PROPOSITION IV.

One plane, and one plane only, can be drawn

1°. *Through a straight line and a point not situated in the given line.*

2°. *Through three points which are not in the same straight line.*

3°. *Through two straight lines which intersect each other.*

4°. *Through two parallel straight lines.*

1. For if a plane be drawn through the given line, and be conceived to revolve round it as an axis, it must, in the course of a complete revolution, pass through the given point, and so assume the position enounced in 1°.

Also, one plane only can answer these conditions; for if we suppose a second plane passing through the same straight line and point, it must have at least two intersections with the first, which is impossible.

2. Join two of the points: this case is then reduced to the last.

3. Take a point in each of the lines which is not the point of intersection: join these two points: the case is now the same as the two former.

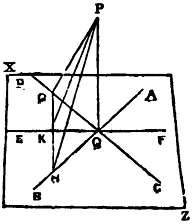
4. Parallel straight lines are, by their definition, in the same plane, and, by the first case, one plane only can be drawn through either of them and a point assumed in the other.

Cor. Hence the position of a plane is determined by

1. *A straight line and a point not in the given straight line.*
2. *A triangle, or three points not in the same straight line.*
3. *Two straight lines which intersect each other.*
4. *Two parallel straight lines.*

PROPOSITION V.

If a straight line be perpendicular to two other straight lines which intersect at its foot in a plane, it will be perpendicular to every other straight line drawn through its foot in the same plane, and will therefore be perpendicular to the plane.



Let XZ be a plane, and let the straight line PQ be perpendicular to the two straight lines AB, CD, which intersect in Q in the plane XZ.

Draw any straight line EF through Q;

Then PQ will be perpendicular to EF.

Draw through any point K in QF a straight line GH, such that GK=KH.

Join P, G; P, K; P, H;

Then, since GH, the base of the Δ GQH, is bisected in K,

$$\therefore GQ^2+HQ^2=2GK^2+2QK^2 \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, since GH, the base of Δ GPH, is bisected in K,

$$\therefore GP^2+HP^2=2GK^2+2PK^2.$$

But the angles PQG, PQH are right angles; ∴ the above becomes

$$PQ^2+GQ^2+PQ^2+HQ^2=2GK^2+2PK^2 \quad . \quad . \quad . \quad (2)$$

Taking (1) from (2), there remains

$$2PQ^2=2PK^2-2QK^2;$$

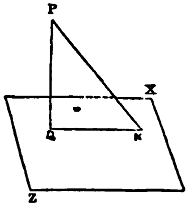
$$\therefore PQ^2+QK^2=PK^2.$$

Hence PQK is a right angle.

In like manner, it may be proved that PQ is at right angles to every other straight line passing through Q in the plane XZ.

PROPOSITION VI.

A perpendicular is the shortest line which can be drawn from any point to a plane.



Let PQ be perpendicular to the plane XZ;

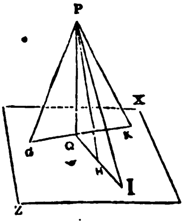
From P draw any other straight line PK to the plane XZ;

Then PQ<PK.

In the plane XZ draw the straight line QK, joining the points Q, K.

Then, since the line PQ is perpendicular to the plane XZ, the angle PQK is a right angle; and ∴ PQ is less than any other line PK (Geom., th. 21, B. I.).

Cor. 1. Hence oblique lines equally distant from the perpendicular are equal; and, if two oblique lines be unequally distant from the perpendicular, the more distant is the larger.

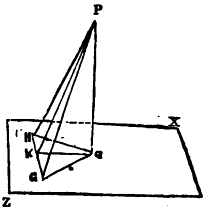


That is, if QG, QH, QK, ... are all equal, then PG, PH, PK, ... are all equal; and if QH be greater than QG, then PH is greater than PG.

Cor. 2. A perpendicular measures the distance of any point from a plane. The distance of one point from another is measured by the straight line joining them, because this is the shortest line which can be drawn from one point to another. So, also, the distance from a point to a line is measured by a perpendicular, because this line is the shortest that can be drawn from the point to the line. In like manner, the distance from a point to a plane must be measured by a perpendicular drawn from the point to the plane, because this is the shortest line that can be drawn from the point to the plane.

PROPOSITION VII.

Let PQ *be a perpendicular on the plane* XZ, *and* GH *a straight line in that plane; if from* Q, *the foot of the perpendicular,* QK *be drawn perpendicular to* GH, *and* P, K *be joined, then* PK *will be perpendicular to* GH.



Take KG=KH; join P, G; P, H; Q, G; Q, H;

∵ KG=KH, and KQ common to the triangles GQK, HQK, and angle GKQ=angle HKQ, each being a right angle,

∴ QG=QH;

∴ PG=PH. Cor. to last Prop.

Hence the two triangles GKP, HKP have the two sides GK, KP equal to the two sides HK, KP, and the remaining side GP equal to the remaining side HP.

∴ Angle GKP=angle HKP; and ∴ each of them is a right angle.

Cor. GH is perpendicular to the plane PQK, for GH is perpendicular to each of the two straight lines KP, KQ.

REMARK.—The two straight lines PQ, GH present an example of two straight lines which do not meet, because they are not situated in the same plane.

The shortest distance between these two lines is the straight line QK, which is perpendicular to each of them.

For, join any other two points, as P, G;

$$\left.\begin{array}{ll}\text{Then,} & PG > PK \\ \text{And,} & KP > KQ\end{array}\right\}\text{ last Prop.;}$$

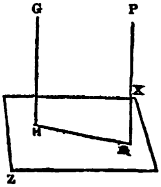
$$\text{Therefore} \quad PG > KQ.$$

The two lines PQ, GH, although not situated in the same plane, are considered to form a right angle with each other. For PQ and a straight line drawn through any point in PQ parallel to GH would form a right angle.

In like manner, PG and QK, which represent any two straight lines not situated in the same plane, are considered to form with each other the same angle which PG would make with any parallel to QK, drawn through a point in PG.

PROPOSITION VIII.

If two straight lines be perpendicular to the same plane, they will be parallel to each other.



Let each of the straight lines PQ, GH be perpendicular to the plane XZ.

Then PQ will be parallel to GH.

In the plane XZ draw the straight line QH, joining the points Q, H.

Then, since PQ, GH are perpendicular to the plane XZ, they are perpendicular to the straight line QH in that plane; and, since PQ, GH are both perpendicular to the same line QH, they are parallel to each other (Geom., th. 14, Cor., B. I.).

Cor. 1. Conversely, if two straight lines be parallel, and if one of them be perpendicular to any plane, the other will also be perpendicular to the same plane.

Cor. 2. Two straight lines parallel to a third are parallel to each other.

For, conceive a plane perpendicular to any one of them, then the other two, being parallel to the first, will be perpendicular to the same plane; hence, by the Prop., they will be parallel to each other.

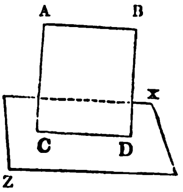
The three straight lines are not supposed to be in the same plane: in this case the proposition has been already demonstrated.

PROPOSITION IX.

If a straight line without a given plane be parallel to a straight line in the plane, it will be parallel to the plane.

Let AB, lying without the plane XZ, be parallel to CD, lying in the plane:

Then AB is parallel to the plane XZ.



Through the parallels AB, CD draw the plane ABCD.

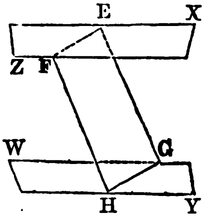
If the line AB can meet the plane XZ, it must meet it in some point of the line CD, which is the common intersection of the two planes.

But AB can not meet CD, because AB is parallel to CD.

Hence AB can not meet the plane XZ; *i. e.*, AB is parallel to the plane XZ.

PROPOSITION X.

The sections made by a plane cutting two parallel planes are parallel.



Let FE, GH be the sections made by the plane GF, which cuts the parallel planes XZ, WY;

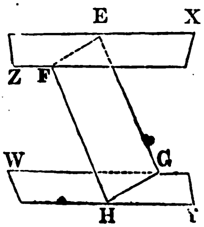
Then FE will be parallel to GH.

For if the lines FE, GH, which are situated in the same plane, be not parallel, they will meet if produced. Therefore the planes XZ, WY, in which these lines lie, will meet if produced, and ∴ can not be parallel, which is contrary to the hypothesis.

∴ FE is parallel to GH.

PROPOSITION XI.

Parallel straight lines included between two parallel planes are equal.



Let the parallels EG, FH be cut by the parallel planes XZ, WY, in the points G, H, E, F;

Then EG = FH.

Through the parallels EG, FH draw the plane EGHF, intersecting the parallel planes in GH, FE.

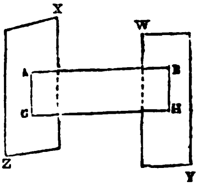
Then GH is parallel to FE, by last Prop.;

And GE is parallel to HF;
∴ GHFE is a parallelogram; and therefore
EG=FH.

Cor. Two parallel planes are every where equidistant.

PROPOSITION XII.

If two planes be parallel to each other, a straight line which is perpendicular to one of the planes will be perpendicular to the other also.



Let the two planes XZ, WY be parallel, and let the straight line AB be perpendicular to the plane XZ;

Then AB will be perpendicular to WY.

For, from any point H in the plane WY draw HG perpendicular to the plane XZ, and draw AG, BH.

Then, since BA, HG are both perpendicular to XZ, ∴ the angles A, G are right angles.

And, since the planes XZ, WY are parallel, ∴ the perpendiculars BA, HG are equal.

Hence AG is parallel to BH, and AB, being perpendicular to AG, is perpendicular to BH also.

In like manner, it may be proved that AB is perpendicular to all other lines which can be drawn from B in the plane WY.

∴ AB is perpendicular to the plane WY.

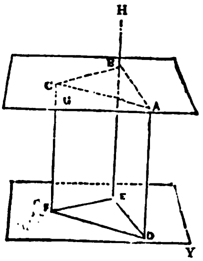
Cor. Conversely, if two planes be perpendicular to the same straight line, they will be parallel to each other.

PROPOSITION XIII.

If two straight lines which form an angle be parallel to two other straight lines which form an angle in the same direction, although not in the same plane with the former, the two angles will be equal, and their planes will be parallel.

Let the two straight lines AB, BC, in the plane XZ, be parallel to the two DE, EF, in the plane WY;

Then angle ABC=angle DEF.



For, make AB=DE, BC=EF; join A, C; D, F; A, D; B, E; C, F;

Then the straight lines AD, BE, which join the equal and parallel straight lines AB, DE, are themselves equal and parallel.

For the same reason, CF, BE are equal and parallel.

∴ AD, CF are equal and parallel, and ∴ AC, DF are also equal and parallel.

Hence the two triangles ABC, DEF, having all their sides equal each to each, have their angles also equal:

∴ angle ABC = angle DEF.

Again, the plane XZ is parallel to the plane WY.

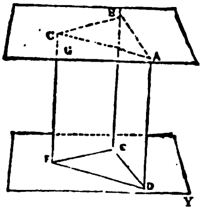
For if not, let a plane drawn through A, parallel to DEF, meet the straight lines FC, EB, in G and H.

Then DA=EH=FG, Prop.

But DA=EB=FC.

Therefore EH=EB, FG=FC,

which is absurd; hence,



Cor. 1. If two parallel planes XZ, WY, are met by two other planes ADEB, CFEB, the angles ABC, DEF, formed by the intersection of the parallel planes, will be equal.

For the section AB is parallel to the section DE, Prop.

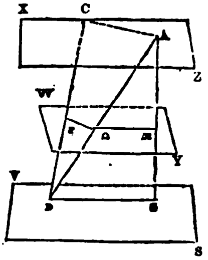
So also the section BC is parallel to the section EF.

∴ angle ABC = angle DEF.

Cor. 2. If three straight lines AD, BE, CF, not situated in the same plane, be equal and parallel, the triangles ABC, DEF, formed by joining the extremities of these straight lines, will be equal, and their planes will be parallel.

PROPOSITION XIV.

If two straight lines be cut by parallel planes, they will be cut in the same ratio.



Let the straight lines AB, CD be cut by the parallel planes XZ, WY, VS, in the points A, E, B, C; F, D;

Then, AE : EB :: CF : FD.

Join A, C; B, D; A, D; and let AD meet the plane WY in G; join E, G; G, F;

Then the intersections EG, BD, of the parallel planes WY, VS, are parallel (Prop. 10).

∴ AE : EB :: AG : GD.

Again, the intersections AC, GF, of the parallel planes XZ, YW, with the plane CG, are parallel;

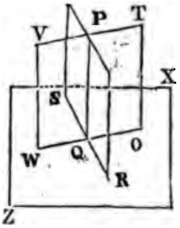
∴ AG : GD :: CF : FD.

∴ comparing this with the first proportion,

AE : EB :: CF : FD.

PROPOSITION XV.

If a straight line be at right angles to a plane, every plane which passes through it will be at right angles to that plane.



Let the straight line PQ be at right angles to the plane XZ.

Through PQ draw any plane PO, intersecting XZ in the line OQW.

Then the plane PO is perpendicular to the plane XZ.

Draw RS in the plane XZ perpendicular to WQO.

Then, since the straight line PQ is perpendicular to the plane XZ, it is perpendicular to the two straight lines RS, OW, which pass through its foot in that plane.

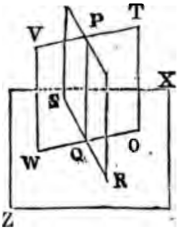
But the angle PQR, contained between PQ, QR, which are perpendiculars to OW, the common intersection of the planes XZ, PO, measures the angle of the two planes (def.

5); hence, since this angle is a right angle, the two planes are perpendicular to each other.

Cor. If three straight lines, such as PQ, RS, OW, be perpendicular to each other, each will be perpendicular to the plane of the two others, and the three planes will be perpendicular to each other.

PROPOSITION XVI.

If two planes be perpendicular to each other, a straight line drawn in one of the planes perpendicular to their common section will be perpendicular to the other plane.



Let the plane VO be perpendicular to the plane XZ, and let OW be their common section.

In the plane VO draw PQ perpendicular to OW;

Then PQ is perpendicular to the plane XZ.

From the point Q draw QR in the plane XZ perpendicular to OW.

Then, since the two planes are perpendicular, the angle PQR is a right angle.

∴ The straight line PQ is perpendicular to the straight lines QR, QO, which intersect at its foot in the plane XZ.

∴ PQ is perpendicular to the plane XZ.

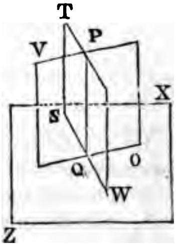
Cor. If the plane VO be perpendicular to the plane XZ, and if from any point in OW, their common intersection, we erect a perpendicular to the plane XZ, that straight line will lie in the plane VO.

For if not, then we may draw from the same point a straight line in the plane VO perpendicular to OW, and this line, by the Prop., will be perpendicular to the plane XZ.

Thus we should have two straight lines drawn from the same point in the plane XZ, each of them perpendicular to the given plane, which is absurd.

PROPOSITION XVII.

If two planes which cut each other be each of them perpendicular to a third plane, their common section will be perpendicular to the same plane.



Let the two planes VO, TW, whose common intersection is PQ, be both perpendicular to the plane XZ.

Then PQ is perpendicular to the plane XZ.

For, from the point Q erect a perpendicular to the plane XZ.

Then, by the Cor. to the last Prop., this straight line must be situated at once in the planes VO and TW, and is therefore their common section.

SOLID ANGLES.

BOOK VI.

DEFINITIONS.

1. A *solid angle* is the angular space contained between several planes which meet in the same point.

2. Three planes, at least, are required to form a solid angle.

3. A solid angle is called a *trihedral*, *tetrahedral*, &c., *angle*, according as it is formed by *three*, *four*, ... plane angles.

4. If three planes intersect each other in a common point, the indefinite space bounded by those three planes is called a *triangular pyramidal space.*

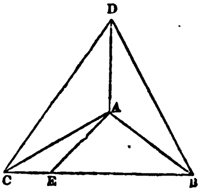
5. If three planes intersect each other in parallel lines, the indefinite space bounded by those three planes is called a *triangular prismatic space.*

6. If four or more planes intersect each other in a common point, the indefinite space bounded by those planes is called a *polyhedral pyramidal space.*

7. If four or more planes intersect each other in parallel edges, the indefinite space bounded by them is called a *polyhedral prismatic space.*

PROPOSITION I.

If a solid angle be contained by three plane angles, the sum of any two of these angles will be greater than the third.



It is unnecessary to demonstrate this proposition except in the case where the plane angle which is compared with the two others is greater than either of them.

Let A be a solid angle contained by the three plane angles BAC, CAD, DAB, and let BAC be the greatest of these angles;

Then, $\text{CAD}+\text{DAB}>\text{BAC}$.

In the plane BAC draw the straight line AE, making the angle $\text{BAE}=\text{angle BAD}$.

Make $\text{AE}=\text{AD}$, and through E draw any straight line BEC, cutting AB, AC, in the points B, C; join D, B; D, C;

Then, $\because \text{AD}=\text{AE}$, and AB is common to the two triangles DAB, BAE, and the angle $\text{DAB}=\text{angle BAE}$,

$$\therefore \text{BD}=\text{BE}.$$

But $$\text{BD}+\text{DC}>\text{BE}+\text{EC},$$

$$\therefore \text{DC}>\text{EC}.$$

Again, $\because \text{AD}=\text{AE}$, and AC is common to the two triangles DAC, EAC, but the base $\text{DC}>\text{base EC}$,

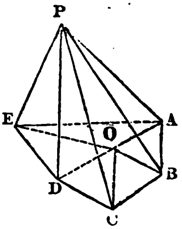
$$\therefore \text{angle DAC}>\text{angle EAC}.$$

But $$\text{angle DAB}=\text{angle BAE}.$$

$$\therefore \text{angle CAD}+\text{angle DAB}>\text{angle BAE}+\text{angle EAC}$$
$$>\text{angle BAC}.$$

PROPOSITION II.

The sum of the plane angles which form a solid angle is always less than four right angles.



Let P be a solid angle contained by any number of plane angles APB, BPC, CPD, DPE, EPA.

Let the solid angle P be cut by any plane ABCDE.

Take any point, O, in this plane; join A, O; B, O; C, O; D, O; E, O;

Then, since the sum of all the angles of every triangle is always equal to two right angles, the sum of all the angles of the triangles ABP, BBC, ... about the point P, will be equal to the sum of all the angles of the equal number of triangles AOB, BOC,...... about the point O.

Again, by the last proposition, angle $\text{ABC}<\text{angle ABP}+\text{angle CBP}$; in like manner, angle $\text{BCD}<\text{angle BCP}+\text{angle DCP}$, and so for all the angles of the polygon ABCDE.

Hence the sum of the angles at the bases of the triangles whose vertex is O is less than the sum of the angles at the bases of the triangles whose vertex is P.

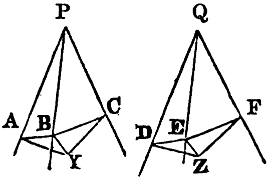
∴ The sum of the angles about the point O must be greater than the sum of the angles about the point P.

But the sum of the angles about the point O is four right angles.

∴ The sum of the angles about the point P is less than four right angles.

PROPOSITION III.

If two solid angles be formed by three plane angles which are equal, each to each, the planes in which these angles lie will be equally inclined to each other.



Let P, Q, be two solid angles, each contained by three plane angles;

Let angle APC=angle DQF, angle APB=angle DQE, and angle BPC=angle EQF.

Then, the inclination of the planes APC, APB, will be equal to the inclination of the planes DQF, DQE.

Take any point, B, in the intersection of the planes APB, CPB.

From B draw BY perpendicular to the plane APC, meeting the plane in Y.

From Y draw YA, YC, perpendiculars on PA, PC; join A, B; B, C;

Again, take QE=PB, from E draw EZ perpendicular to the plane DQF, meeting the plane in Z; from Z draw ZD, ZF, perpendiculars on QD, QF; join D, E; E, F.

The triangle PAB is right angled at A, and the triangle QDE is right angled at D. (Geom. of Planes, Prop. 7.)

Also, the angle APB=angle DQE, by construction.

∴ angle PBA=angle QED.

But the side PB=side QE, ∴ the two triangles APB, DQF, are equal and similar.

∴ PA=QD, and AB=DE.

In like manner, we can prove that

PC=QF, and BC=EF.

We can now prove that the quadrilateral PAYC is equal to the quadrilateral QDZF.

For, let the angle APC be placed upon the equal angle DQF, then the point A will fall upon the point D, and the point C on the point F, because PA=QD, and PC=QF.

At the same time, AY, which is perpendicular to PA, will fall upon DZ, which is perpendicular to QD; and, in like manner, CY will fall upon FZ.

Hence the point Y will fall on the point Z, and we shall have

$$AY=DZ, \text{ and } CY=FZ.$$

But the triangles AYB, DZE, are right angled at Y and Z, the hypothenuse AB=hypothenuse DE, and the side AY =side DZ; hence these two triangles are equal.

$$\therefore \text{angle } YAB=\text{angle } ZDE.$$

The angle YAB is the inclination of the planes APC, APB; and

The angle ZDE is the inclination of the planes DQF, DQE.

∴ these planes are equally inclined to each other.

In the same manner, we prove that angle YCB=angle ZFE, and, consequently, the inclination of the planes APC, BPC, is equal to the inclination of the planes DQF, EQF.

We must, however, observe that the angle A of the right-angled triangle YAB is not, properly speaking, the inclination of the two planes APC, APB, except when the perpendicular BY falls upon the same side of PA as PC does; if it fall upon the other side, then the angle between the two planes will be obtuse, and, added to the angle A of the triangle YAB, will make up two right angles. But, in this case, the angle between the two planes DQF, DQE, will also be obtuse, and, added to the angle D of the triangle ZDE, will make up two right angles.

Since, then, the angle A will always be equal to the angle D, we infer that the inclination of the two planes APC, APB, will always be equal to the inclination of the two planes DQF, DQE. In the first case, the inclination of the plane is the angle A or D; in the second case, it is the supplement of those angles.

Scholium.—If two solid trihedral angles have the three plane angles of the one equal to the three plane angles of the other, each to each, and, at the same time, the corresponding angles *arranged in the same manner* in the two solid angles, then these two solid angles will be equal; and if placed one upon the other, they will coincide. In fact, we have already seen that the quadrilateral PAYC will coincide with the quadrilateral QDZF. Thus the point Y falls upon the point Z, and, in consequence of the equality of the triangles AYB, DZE, the straight line YB, perpendicular to the plane APC, is equal to the straight line ZE, perpendicular to the plane DQE; moreover, these perpendiculars lie in the same direction; hence the point B will fall upon the point E, the straight line PB on the straight line QE, and the two solid angles will entirely coincide with each other.

This coincidence, however, can not take place except we suppose the equal plane angles to be *arranged in the same manner* in the two solid angles; for, if the equal planes be *arranged in an inverse order*, or, which comes to the same thing, if the perpendiculars YB, ZE, instead of being situated both on the same side of the planes APC, DQF, were situated on opposite sides of these planes, then it would be impossible to make the two solid angles coincide with each other. It would not, however, be less true, according to the above theorem, that the planes in which the equal angles lie would be equally inclined to each other; so that the two solid angles would be equal in all their constituent parts, without admitting of superposition. This species of equality, which is not absolute, or *equality of coincidence*, has received from Legendre a particular description. He terms it *equality of symmetry*.

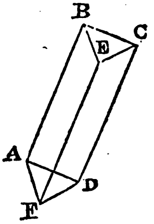
Thus the two solid trihedral angles in question, which have the three plane angles of the one equal to the three plane angles of the other, each to each, but *arranged in an inverse order*, are termed *angles equal by symmetry*, or simply *symmetrical angles*.

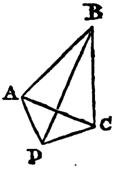
The same observation applies to solid angles formed by more than three plane angles. Thus a solid angle formed by the plane angles A, B, C, D, E, and another solid angle formed by the same angles in an inverse order, A, E, D, C, B, may be such that the planes in which the equal angles are situated are equally inclined to each other. These two solid angles, which would in this case be equal, although not admitting of superposition, would be termed *solid angles equal by symmetry*, or *symmetrical solid angles*.

In plane figures there is no species of equality to which this designation can belong, for all those cases to which the term might seem to apply are cases of *absolute equality*, or *equality of coincidence*. The reason of this is that the position of a plane figure may be altered at pleasure, and one may take the upper part for the under, and *vice versa*. This, however, does not hold in solids, in which the third dimension may be taken in two different directions.

PROPOSITION IV.

If three planes intersect each other in three lines, these lines will either meet in the same common point, or they will be parallel to each other.





Let AE, FC, and BD be three planes intersecting each other respectively in the lines EF, AB, and DC; then each pair of these lines will be in the same plane. Now each pair of these lines must be either parallel or must intersect in a common point. (Prop. 4.)

1°. If two of these lines of intersection, as AB and CB, meet in a point B, that point is in the line AB, and also in the planes ADB and ACB; but the point B is, at the same time, a point of the line CB, and also in both planes CDB, ACB. The point B is therefore in all the three planes, ADB, DBC, ABC, and is also one of the points common to the two planes BDC, ADB, being a point of the line DB. This third line, DB, therefore, meets the same point, B, as the lines AB, CB do; that is, all these lines meet in the same point.

2°. But if the two lines of intersection, AB, FE, be parallel to each other, the third line, DC, can never meet either of these lines, because, if it can meet either of them, then the other must also intersect at the same point (Dem. 1), and the first two not parallel, which is contrary to the supposition.

Again, as every two of these lines of intersection are lines of the same plane, and can not meet, therefore they must all be parallel to each other.

SOLID GEOMETRY.

BOOK VII.

DEFINITIONS.

1. Similar solid figures are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes.

2. A pyramid is a solid figure contained by planes that are constituted between one plane and one point above it, in which they meet.

3. A prism is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to each other, and the others are parallelograms.

4. A sphere is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved.

Thus the inner side of the semicircle revolving round the diameter, which remains fixed, generates a sphere.

5. The axis of a sphere is the fixed right line about which the semicircle revolves.

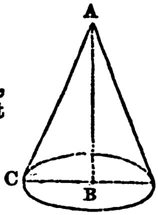
6. The centre of a sphere is the same with that of the semicircle.

7. The diameter of a sphere is any right line which passes through the centre, and is terminated both ways by the superficies of the sphere.

8. A right cone is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone; if it be less than the other side, an obtuse-angled; and if greater, an acute-angled cone.

Thus the side AC, revolving round AB, one of the sides which contains the right angle and remains fixed, generates a cone.



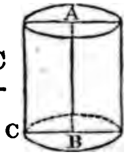
9. The axis of a cone is the fixed right line about which the triangle revolves.

In the preceding figure, AB is the axis.

10. The base of a cone is the circle described by that side containing the right angle which revolves.

11. A cylinder is a solid figure described by the revolution of a right-angled parallelogram about one of its sides, which remains fixed.

Thus the revolution of the parallelogram AC about its side AB, which remains fixed, generates a cylinder.

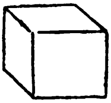


12. The axis of a cylinder is the fixed right line about which the parallelogram revolves.

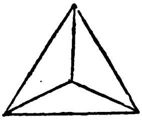
13. The bases of a cylinder are the circles described by the two revolving opposite sides of the parallelogram.

14. Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.

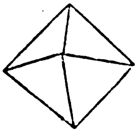
15. A cube is a solid figure contained by six equal squares.



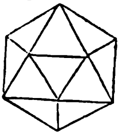
16. A tetrahedron is a solid figure contained by four equal and equilateral triangles.



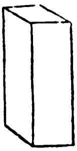
17. An octahedron is a solid figure contained by eight equal and equilateral triangles.



18. A dodecahedron is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.



19. An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

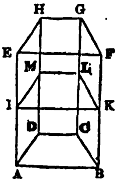


20. A parallelopiped is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.

PROPOSITIONS.

PROPOSITION I.

If a prism be cut by a plane parallel to its base, the section will be equal to and like the base.

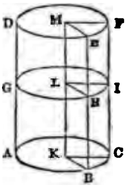


Let AG be any prism, and IL a plane parallel to the base AC, then will the plane IL be equal to and like the base AC, or the two planes will have all their sides and all their angles equal.

For the two planes, AC, IL, being parallel, by hypothesis; and two parallel planes, cut by a third plane, having parallel sections; therefore IK is parallel to AB, KL to BC, LM to CD, and IM to AD. But AI and BK are parallels, by Def. 3; consequently, AK is a parallelogram, and the opposite sides, AB, IK, are equal. In like manner, it is shown that KL is=BC, and LM=CD, and IM=AD, or the two planes, AC, IL, are mutually equilateral. But these two planes, having their corresponding sides parallel, have the angles contained by them also equal; namely, the angle A=the angle I, the angle B=the angle K, the angle C=the angle L, and the angle D=the angle M. So that the two planes, AC, IL, have all their corresponding sides and angles equal, or are equal and like. Q. E. D.

PROPOSITION II.

If a cylinder be cut by a plane parallel to its base, the section will be a circle equal to the base.



Let AF be a cylinder, and GHI any section parallel to the base ABC; then will GHI be a circle equal to ABC.

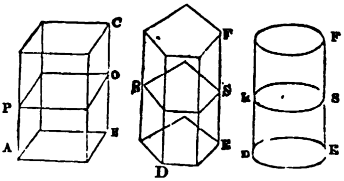
For, let the planes KE, KF, pass through the axis of the cylinder MK, and meet the section GHI in the three points H, I, L, and join the points as in the figure.

Then, since KL, CI, are parallel, and the plane KI, meeting the two parallel planes ABC, GHI, makes the two sections KC, LI, parallel; the figure KLIC is therefore a parallelogram, and, consequently, has the opposite sides LI, KC equal, where KC is a radius of the circular base.

In like manner, it is shown that LH is equal to the radius KB; and that any other lines, drawn from the point L to the circumference of the section GHI, are all equal to radii of the base; consequently, GHI is a circle, and equal to ABC. Q. E. D.

PROPOSITION III.

All prisms, and a cylinder of equal bases and altitudes, are equal to each other.



Let AC, DF, be two prisms, and a cylinder upon equal bases AB, DE, and having equal altitudes; then will the solids AC, DF, be equal.

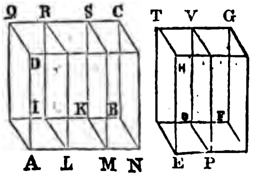
For, let PQ, RS, be any two sections parallel to the bases, and equidistant from them. Then, by the last two propositions, the section PQ is equal to the base AB, and the section RS equal to the base DE. But the bases AB, DE, are equal by the hypothesis; therefore the sections PQ, RS, are also equal. And, in like manner, it may be shown that any other corresponding sections are equal to one another.

Since, then, every section in the prism AC is equal to its corresponding section in the prism, or cylinder RS, the prisms and cylinders themselves, which are composed of those sections, must also be equal. Q. E. D.

Corol. Every prism or cylinder is equal to a rectangular parallelopipedon of an equal base and altitude.

PROPOSITION IV.

Rectangular parallelopipedons of equal altitudes have to each other the same proportion as their bases.



Let AC, EG, be two rectangular parallelopipedons, having the equal altitudes AD, EH; then will AC be to EG as the base AB is to the base EF.

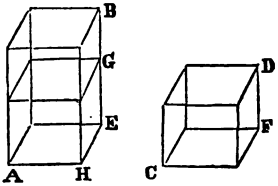
For, let the proportion of the base AB to the base EF be that of any one number m (3) to any other number n (2); and conceive AB to be divided into m equal parts or rectangles, AI, LK, MB (by dividing AN into that number of equal parts, and drawing IL, KM, parallel to BN). And let EF be divided, in like manner, into n equal parts or rectangles, EO, PF: all of these parts of both bases being mutually equal among themselves. And through the lines of division let the plane sections LR, MS, PV pass parallel to AQ, ET.

Then the parallelopipedons AR, LS, MC, EV, PG, are all equal, having equal bases and heights. Therefore the solid AC is to the solid EG as the number of parts in AC to the number of equal parts in EG, or as the number of parts in AB to the number of equal parts in EF; that is, as the base AB to the base EF. Q. E. D.

Corol. From this proposition, and the corollary to the last, it appears that all prisms and cylinders of equal altitudes are to each other as their bases, every prism and cylinder being equal to a rectangular parallelopipedon of an equal base and height.

PROPOSITION V.

Rectangular parallelopipedons of equal bases are in proportion to each other as their altitudes.



Let AB, CD, be two rectangular parallelopipedons standing on the equal bases AE, CF; then will AB be to CD as the altitude EB is to the altitude DF.

For, let AG be a rectangular parallelopipedon on the base AE, and its altitude EG equal to the altitude FD of the solid CD.

Then AG and CD are equal, being prisms of equal bases and altitudes. But if HB, HG, be considered as bases, the solids AB, AG, of equal altitude AH, will be to each other as those bases HB, HG. But these bases HB, HG, being parallelograms of equal altitude HE, are to each other as their bases EB, EG; and therefore the two prisms AB, AG, are to each other as the lines EB, EG. But AG is equal CD, and EG equal FD; consequently, the prisms AB, CD, are to each other as their altitudes EB, FD; that is, AB : CD :: EB : FD. Q. E. D.

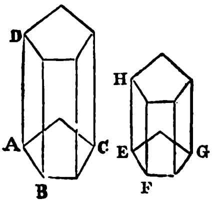
Corol. 1. From this proposition, and the corollary to Prop. 3, it appears that all prisms and cylinders of equal bases are to one another as their altitudes.

Corol. 2. Because, by Corol. 1, prisms and cylinders are as their altitudes when their bases are equal; and, by the corollary to the last theorem, they are as their bases when their altitudes are equal; therefore, universally, when neither are equal, they are to one another as the product of their bases and altitudes. And hence, also, these products are the proper numeral measures of their quantities or magnitudes.

PROPOSITION VI.

Similar prisms and cylinders are to each other as the cubes of their altitudes, or of any other like linear dimensions.

Let ABCD, EFGH, be two similar prisms; then will

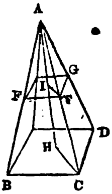


the prism CD be to the prism GH as AB^3 to EF^3, or as AD^3 to EH^3.

For the solids are to each other as the product of their bases and altitudes (Prop. 5, Cor. 2); that is, as AC . AD to EG . EH. But the bases, being similar planes, are to each other as the squares of their like sides; that is, AC to EG as AB^2 to EF^2; therefore the solid CD is to the solid GH as AB^2 . AD to EF^2 . EH. But BD and FH, being similar planes, have their like sides proportional; that is, AB : EF :: AD : EH, or AB^2 : EF^2 :: AD^2 : EH^2; therefore AB^2 . AD : EF^2 . EH :: AD^3 : EH^3, or :: AB^3 : EF^3; and, consequently, the solid CD : solid GH :: AB^3 : EF^3 :: AD^3 : EH^3. Q. E. D.

PROPOSITION VII.

In a pyramid, a section parallel to the base is similar to the base, and these two planes will be to each other as the squares of their distances from the vertex.



Let ABCD be a pyramid, and EFG a section parallel to the base BCD; also AIH a line perpendicular to the two planes at H and I; then will BD, EG, be two similar planes, and the plane BD will be to the plane EG as AH^2 to AI^2.

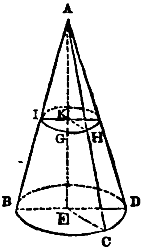
For, join CH, FI. Then, because a plane cutting two parallel planes makes parallel sections, therefore the plane ABC, meeting the two parallel planes BD, EG, makes the sections BC, EF, parallel; in like manner, the plane ACD makes the sections CD, FG, parallel. Again, because two pair of parallel lines make equal angles, the two EF, FG, which are parallel to BC, CD, make the angle EFG equal the angle BCD. And, in like manner, it is shown that each angle in the plane EG is equal to each angle in the plane BD, and, consequently, those two planes are equiangular.

Again, the three lines AB, AC, AD, making with the parallels BC, EF, and CD, FG, equal angles; and the angles at A being common, the two triangles ABC, AEF, are equiangular, as also the two triangles ACD, AFG, and have therefore their like sides proportional, namely, AC : AF :: BC : EF :: CD : FG. And, in like manner, it may be shown that all the lines in the plane EG are proportional to all the corresponding ones in the base BD. Hence these two planes, having their angles equal and their sides proportional, are similar.

But similar planes being to each other as the squares of their like sides, the plane BD : EG :: $BC^2 : EF^2$: or :: $AC^2 : AF^2$, by what is shown above. But the two triangles AHC, AIF, having the angles H and I right ones, and the angle A common, are equiangular, and have therefore their like sides proportional, namely, AC : AF :: AH : AI, or $AC^2 : AF^2 :: AH^2 : AI^2$. Consequently, the two planes BD, EG, which are as the former squares AC^2, AF^2, will be also as the latter squares AH^2, AI^2; that is, BD : EG :: $AH^2 : AI^2$.

PROPOSITION VIII.

In a right cone, a section parallel to the base is a circle; and this section is to the base as the squares of their distances from the vertex.



Let ABCD be a right cone, and GHI a section parallel to the base BCD; then will GHI be a circle, and BCD, GHI, will be to each other as the squares of their distances from the vertex.

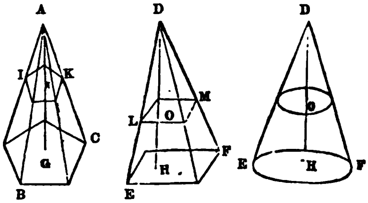
For, draw AKE perpendicular to the two parallel planes; and let the planes ACE, ADE, pass through the axis of the cone AKE, meeting the section in the three points H, I, K.

Then, since the section GHI is parallel to the base BCD, and the planes CK, DK, meet them, HK is parallel to CE, and IK to DE. And, because the triangles formed by these lines are equiangular, KH : EC :: AK : AE :: KI : ED. But EC is equal to ED, being radii of the same circle; therefore KI is also equal to KH. And the same may be shown of any other lines drawn from the point K to the circumference of the section GHI, which is therefore a circle.

Again, by similar triangles, AK : AE :: KH : EC; hence $AK^2 : AE^2 :: KH^2 : EC^2$; but $KH^2 : EC^2$ :: circle GHI : circle BCD; therefore $AK^2 : AE^2$ :: circle GHI : circle BCD. Q. E. D.

PROPOSITION IX.

All pyramids and right cones of equal bases and altitudes are equal to one another.



Let ABC, DEF, be any pyramids and cone of equal bases BC, EF, and equal altitudes AG, DH; then will the pyramids and cone ABC and DEF be equal.

For, parallel to the bases, and at equal distances, AN, DO, from the vertices, suppose the planes IK, LM, to be drawn.

Then, by propositions 7 and 8,

$$DO^2 : DH^2 :: LM : EF, \text{ and}$$
$$AN^2 : AG^2 :: IK : BC.$$

But, since AN^2, AG^2, are equal to DO^2, DH^2; therefore IK : BC :: LM : EF. But BC is equal to EF, by hypothesis; therefore IK is also equal to LM.

In the same manner, it is shown that any other sections, at equal distance from the vertex, are equal to each other.

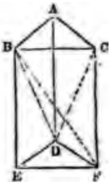
Since, then, every section in the cone is equal to the corresponding section in the pyramids, and the heights are equal, the solids ABC, DEF, composed of those sections, must be equal also. Q. E. D.

PROPOSITION X.

Every pyramid of a triangular base is the third part of a prism of the same base and altitude.

Let ABCDEF be a prism, and BDEF a pyramid, upon the same triangular base DEF; then will the pyramid BDEF be a third part of the prism ABCDEF.

For, in the planes of the three sides of the prism, draw



the diagonals BF, BD, CD. Then the two planes BDF, BCD, divide the whole prism into the three pyramids BDEF, DABC, DBCF, which are proved to be all equal to one another as follows:

Since the opposite ends of the prism are equal to each other, the pyramid whose base is ABC and vertex D is equal to the pyramid whose base is DEF and vertex B (Prop. 9), being pyramids of equal base and altitude.

But the latter pyramid, whose base is DEF and vertex B, is the same solid as the pyramid whose base is BEF and vertex D, and this is equal to the third pyramid, whose base is BCF and vertex D, being pyramids of the same altitude and equal bases BEF, BCF.

Consequently, all the three pyramids which compose the prism are equal to each other, and each pyramid is the third part of the prism, or the prism is triple of the pyramid. Q. E. D.

Corol. 1. Every pyramid, whatever its figure may be, is the third part of a prism of the same base and altitude; since the base of the prism, whatever be its figure, may be divided into triangles, and the whole solid into triangular prisms and pyramids.

Corol. 2. Any right cone is the third part of a cylinder or of a prism of equal base and altitude, since it has been proved that a cylinder is equal to a prism, and a cone equal to a pyramid of equal base and altitude.

Corol. 3. Every triangular prism may be divided into three equal triangular pyramids of the same base and altitude with the prism.

SCHOLIUM.—Whatever has been demonstrated of the proportionality of prisms or cylinders holds equally true of pyramids or cones, the former being always triple the latter, viz., that similar pyramids or cones are as the cubes of their like linear sides, or diameters, or altitudes, &c.

PROPOSITION XI.

If a sphere be cut by a plane, the section will be a circle.

Because the radii of the sphere are all equal, each of them being equal to the radius of the describing semicircle, it is evident that if the section pass through the centre of the sphere, then the distance from the centre to every point in the periphery of that section will be equal to the radius of the sphere, and the section will therefore be a circle of the same radius as the sphere. But if the plane do not pass through the centre, draw a perpendicular to it from the centre, and draw any number of radii of the sphere to the intersection of its surface with the plane; then these radii are evidently the hypothenuses of a corresponding number of right-angled triangles, which have the perpendicular from the centre on the plane of the section as a common side; consequently, their other sides are all equal, and therefore the section of the sphere by the plane is a circle, whose centre is the point in which the perpendicular cuts the plane.

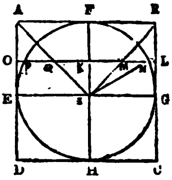
Corol. If two spheres intersect one another, the common section is a circle.

SCHOLIUM.—All the sections through the centre are equal to one another, and are greater than any other section which does not pass through the centre. Sections through the centre are called *great circles*, and the other sections *small* or *less circles*. Also a straight line drawn through the centre of a circle of the sphere perpendicular to the plane of the circle is a diameter of the sphere, and the extremities of this diameter are called the *poles* of the circle. Hence it is evident that the arcs of great circles between the pole and circumference are equal, for the chords drawn in the sphere from either pole of a circle to the circumference are all equal.

PROPOSITION XII.

Every sphere is two thirds of its circumscribing cylinder.

Let ABCD be a cylinder circumscribing the sphere EFGH; then will the sphere EFGH be two thirds of the cylinder ABCD. For, let the plane AC be a section of the sphere and cylinder through the centre I, and join AI, BI. Let FIH be parallel to AD or BC, and EIG and KL parallel to AB or DC, the base of the cylinder; the latter line KL meeting BI in M, and the circular section of the sphere in N.



Then, if the whole plane HFBC be conceived to revolve about the line HF as an axis, the square FG will describe a cylinder AG, and the quadrant IFG will describe a hemisphere EFG, and the triangle IFB will describe a cone IAB. Also, in the rotation, the three lines or parts, KL, KN, KM, as radii, will describe corresponding circular sections of these solids, viz., KL a section of the cylinder, KN a section of the sphere, and KM a section of the cone.

Now FB being equal to FI or IG, and KL parallel to FB, then, by similar triangles, $IK = KM$ (B. III., th. 4), and IKN is a right-angled triangle; hence IN^2 is equal to $IK^2 + KN^2$ (B. I., th. 34). But KL is equal to the radius IG or IN, and $KM = IK$; therefore KL^2 is equal to $KM^2 + KN^2$, or the square of the longest radius of the said circular sections is equal to the sum of the squares of the two others. Now circles are to each other as the squares of their diameters or of their radii; therefore the circle described by KL is equal to both the circles described by KM and KN, or the section of the cylinder is equal to both the corresponding sections of the sphere and cone. And as this is always the case in every parallel position of KL, it follows that the cylinder EB, which is composed of all the former sections, is equal to the hemisphere EFG and cone IAB, which are composed of all the latter sections.

But the cone IAB is a third part of the cylinder EB (Prop. 10, Corol. 2); consequently, the hemisphere EFG is equal to the remaining two thirds, or the whole sphere EFGH is equal to two thirds of the whole cylinder ABCD.

Corol. 1. A cone, hemisphere, and cylinder of the same base and altitude are to each other as the numbers 1, 2, 3.

Corol. 2. All spheres are to each other as the cubes of their diameters, all these being like parts of their circumscribing cylinders.

Corol. 3. From the foregoing demonstration, it appears that the spherical zone or frustum EGNP is equal to the difference between the cylinder EGLO and the cone IMQ, all of the same common height IK. And that the spherical segment PFN is equal to the difference between the cylinder ABLO and the conic frustum AQMB, all of the same common altitude FK.

SCHOLIUM.—By the scholium to Prop. 10, we have

cone AIB : cone QIM $:: IF^3 : IK^3 :: FH^3 : (FH-2FK)^3$;

$\therefore$ cone AIB : frustum ABMQ $:: FH^3 : FH^3-(FH-2FK)^3$

$:: FH^3 : 6FH^2FK-12FH.FK^2+8FK^3$;

but cone AIB$=$one third of the cylinder ABGE; hence

cylinder AG : frustum ABMQ $:: 3FH^3 : 6FH^2.FK-12FH.FK^2+8EK^3$.

Now cylinder AL : cylinder AG :: FK : FI;

$\therefore$ cylinder AL : frustum ABMQ $:: 6FH^2 : 6FH^2-12FH.FK+8FK^2$;

$\therefore$ cylinder AL : segment PFN $:: 6FH^2 : 12FH:FK-8FK^2$;

dividendo $:: \frac{3}{2}FH^2 : FK(3FH-2FK)$.

But cylinder AL$=$circular base whose diameter is AB or FH multiplied by the height FK; hence cylinder AL$=$circle EFGH$\times$FK.

$$\therefore \text{segment PFN}=\frac{2}{3}.\frac{\text{circle EFGH}}{FH^2}(3FH-2FK)FK^2.$$

SPHERICAL GEOMETRY.

BOOK VIII.

DEFINITIONS.

1. A SPHERE is a solid terminated by a curve surface, and is such that all the points of the surface are equally distant from an interior point, which is called the *centre* of the sphere.

We may conceive a sphere to be generated by the revolution of a semicircle about its diameter; for the surface described by the motion of the curve will have all its points equally distant from the centre.

2. The *radius* of a sphere is a straight line drawn from the centre to any point on the surface.

The *diameter* or *axis* of a sphere is a straight line drawn through the centre, and terminated both ways by the surface.

It appears from Definition 1 that all the radii of the same sphere are equal, and that all the diameters are equal, and each double of the radius.

3. It will be demonstrated (Prop. 1) that every section of a sphere made by a plane is a circle; this being assumed,

A *great circle* of a sphere is the section made by a plane passing through the centre of the sphere.

A *small circle* of a sphere is the section made by a plane which does not pass through the centre of the sphere.

4. The *pole* of a circle of a sphere is the point on the surface of the sphere equally distant from all the points in the circumference of that circle.

It will be seen (Prop. 2) that all circles, whether great or small, have two poles.

5. A *spherical triangle* is the portion of the surface of a sphere included by the arcs of three great circles.

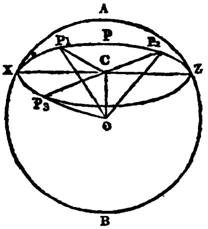
6. These arcs are called the *sides* of the triangle, and each is supposed to be less than half of the circumference.

7. The *angles* of a spherical triangle are the angles contained between the planes in which the sides lie.

8. A plane is said to be a tangent to a sphere when it contains only one point in common with the surface of the sphere.

PROPOSITION I.

Every section of a sphere made by a plane is a circle



Let AZBX be a sphere whose centre is O.

Let XPZ be a section made by the plane XZ.

From O draw OC perpendicular to the plane XZ.

In XPZ take any points P_1, P_2, P_3, ...

Join CP_1, CP_2, CP_3; .... also OP_1, OP_2, OP_3; ....

Then, since OC is perpendicular to the plane HZ, it will be perpendicular to all straight lines passing through its foot in that plane. (Geometry of Planes.)

Hence the angles OCP_1, OCP_2, OCP_3, ... are right angles;

$$\therefore OP_1^2 = CP_1^2 + OC^2;$$
$$OP_2^2 = CP_2^2 + OC^2;$$
$$OP_3^2 = CP_3^2 + OC^2.$$

But, since P_1, P_2, P_3, ... are all points upon the surface of the sphere, ∴ by Def. 1, $OP_1 = OP_2 = OP_3 = \ldots\ldots$

$$\therefore CP_1 = CP_2 = CP_3 \ldots\ldots$$

Hence XPZ is a circle whose centre is C, and every other section of a sphere made by a plane may, in like manner, be proved to be a circle.

Corol. 1. If the plane pass through the centre of the sphere, then $OC = 0$, and the radius of the circle will be equal to the radius of the sphere.

Corol. 2. Hence all great circles are equal to one another, since the radius of each is equal to the radius of the sphere.

Corol. 3. Hence, also, two great circles always bisect each other, for their common intersection passing through the centre is a diameter.

Corol. 4. The centre of a small circle and that of the sphere are in a straight line, which is perpendicular to the plane of the small circle.

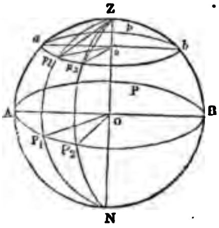
Corol. 5. We can always draw one, and only one, great circle through any two points on the surface of a sphere, for the two given points and the centre of the sphere give three points, which determine the position of a plane.

If, however, the two given points are the extremities of a diameter, then these two points and the centre of the sphere are in the same straight line, and an infinite number of great circles may be drawn through the two points.

Distances on the surface of a sphere are measured by the arcs of great circles. The reason for this is that the shortest line which can be drawn upon the surface of a sphere between any two points is the arc of a great circle joining them.

PROPOSITION II.

If a diameter be drawn perpendicular to the plane of a great circle, the extremities of the diameter will be the poles of that circle, and of all the small circles whose planes are parallel to it.



Let APB be a great circle of the sphere whose centre is O.

Draw ZN a diameter perpendicular to the plane of circle APB.

Then Z and N, the extremities of this diameter, are the poles of the great circle APB, and all the small circles, such as *apb*, whose planes are parallel to that of APB.

Take any points P_1, P_2, ... in the circumference of APB.

Through each of these points respectively, and the points Z and N, describe great circles, ZP_1N, ZP_2N.

Join OP_1, OP_2, ...

Then, since ZO is perpendicular to the plane of APB, it is perpendicular to all the straight lines OP_1, OP_2 ...... drawn through its foot in that plane.

Hence all the angles ZOP_1, ZOP_2, ... are right angles, and $\therefore$ the arcs ZP_1, ZP_2, ... are quadrants.

Thus it appears that the points Z and N are equally distant from all the points in the circumference of APB, and are $\therefore$ the poles of that great circle.

Again, since ZO is perpendicular to the plane APB, it is also perpendicular to the plane *apb*, which is parallel to the former.

Hence the oblique lines Zp_1, Zp_2, ... drawn to p_1, p_2, in the circumference of *apb*, will be equal to each other. (Geometry of Planes.)

$\therefore$ The chords Zp_1, Zp_2, ... being equal, the arcs Zp_1, Zp_2, ... which they subtend, will also be equal.

$\therefore$ The point Z is the pole of the circle *apb*; and, for the same reason, the point N is also a pole.

Corol. 1. Every arc P_1Z drawn from a point in the circumference of a great circle to its pole is a quadrant, and this arc P_1Z makes a right angle with the arc AP_1B. For, the straight line ZO being perpendicular to the plane APB, every plane which passes through this straight line will be perpendicular to the plane APB (Geometry of Planes); hence the angle between these planes is a right angle, or, by (Def. 7), the angle of the arcs AP_1 and Z_1 is a right angle.

Corol. 2. In order to find the pole of a given arc AP_1, of a great circle, take P_1Z equal to a quadrant, and perpendicular to AP_1, the point Z will be a pole of the arc AP_1; or from the points A and P_1 draw two arcs AZ and P_1Z perpendicular to AP_1, the point Z in which they meet is a pole of AP_1.

Corol. 3. Reciprocally, if the distance of the point Z from each of the points A and P_1 is equal to a quadrant, then the point Z is the pole of AP_1, and each of the angles ZAP_1, ZP_1A, is a right angle.

For, let O be the centre of the sphere, draw the radii OA, OP_1, OZ;

Then, since the angles AOZ, P_1OZ, are right angles, the straight line OZ is perpendicular to the straight lines OA,

$\dot{O}P_1$, and is ∴ perpendicular to their plane; hence, by Prop., the point Z is the pole of AP_1, and ∴ the angles ZAP_2, ZP_1A, are right angles.

Corol. 4. Great circles, such as ZA, ZP_1, whose planes are at right angles to the plane of another great circle, as APB, are called its *secondaries;* and it appears from the foregoing corollaries that,

1. The planes of all secondaries pass through the axis, and their circumferences through the poles of their primary; and that the poles of any great circle may always be determined by the intersection of any two of its secondaries.

2. The arcs of all secondaries intercepted between the primary and its poles are = 90°.

3. A secondary bisects all circles parallel to its primary.

Corol. 5. Let the radius of the sphere = R, radius of small circle = r. Distance of two circles, or $Oo = \delta$.

Join Op_1, arc $P_1p_1 = \phi$.

$$R^2 = r^2 + \delta^2.$$
$$r = R \cos. \phi.$$
$$\delta = R \sin. \phi.$$

Corol. 6. Two secondaries intercept similar arcs of circles parallel to their primary, and these arcs are to each other as the cosines of the arcs of the secondaries between the parallels and the primary.

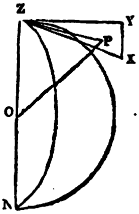
For the arcs of the parallels subtend at their respective centres angles equal to the inclinations of the planes of the secondaries, and these arcs will therefore be similar.

Also, if r_1, r_2, be the radii of two small parallels, the rest of notation as before,

$$\frac{\text{circumference } p_1\, p_2}{\text{circumference } q_1\, q_2} = \frac{\text{whole circumference of 1st}}{\text{whole circumference of 2d}};$$
$$= \frac{r_1}{r_2}.$$
$$= \frac{R \cos. \phi}{R \cos. \phi'}.$$
$$= \frac{\cos. \phi}{\cos. \phi'}.$$

PROPOSITION III.

Every plane perpendicular to a radius at its extremity is a tangent to the sphere in that point.



Let ZXY be a plane perpendicular to the radius OZ.

Then ZXY touches the sphere in Z.

Take any point P in the plane; join ZP; OP;

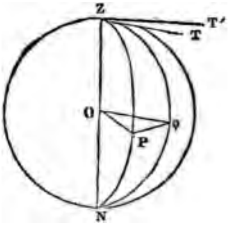
Then, since OZP is a right-angled triangle, ∴ the side OP is$>$side OZ.

Hence the point P is without the sphere; and, in like manner, it may be shown that every point in XYZ, except Z, is without the sphere.

Therefore the plane XYZ is a tangent to the sphere.

PROPOSITION IV.

The angles formed by two arcs of great circles is equal to the angle contained by the tangents drawn to these arcs at their point of intersection, and is measured by the arc described from their point of intersection or pole, intercepted by the arcs containing the angle.



Let ZPN, ZQN, arcs of great circles, intersect in Z.

Draw ZT, ZT′, tangents to the arcs at the point Z.

With Z as pole, describe the arc PQ.

Take O the centre of the sphere, and join OP, OQ.

Then the spherical angle PZQ is equal to the angle TZT′, and is measured by the arc PQ.

For the tangent ZT, drawn in the plane ZPN, is perpendicular to radius OZ.

And the tangent ZT′ drawn in the plane ZQN is perpendicular to radius OZ.

Hence the angle TZT′ is equal to the angle contained by these two planes; that is, to the spherical angle PZQ. (Geometry of Planes.)

Again, since the arcs ZP, ZQ, are each of them equal to a quadrant,

∵ Each of the angles ZOP, ZOQ, is a right angle;

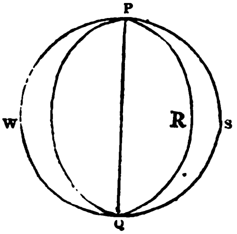
∴ The angle QOP is the angle contained by the planes ZPN, ZQN, and is$=TZT'$.

∴ The arc PQ, which measures the angle POQ, measures the angle between the planes; that is, the spherical angle PZQ.

Corol. 1. The angle under two great circles is measured by the distance between their poles. For the axis of the great circles drawn through their poles being perpendicular to the planes of the circles, the angles under these axes will be equal to the angle between the circles; but the angle under the axes is obviously measured by the arc which joins their extremities; that is, by the distance between their poles.

Corol. 2. The angle under two great circles is measured by the arc of a common secondary intercepted between them.

For, since the secondary passes through the poles of both, taking away from the equal quadrants of the secondary between each circle and its pole, the common arc intercepted between one circle and the pole of the other, the remainders are the intercept of the common secondary between the two circles and the distance between their poles, and these are therefore equal. But the latter is, by the last corollary, the measure of the angle.



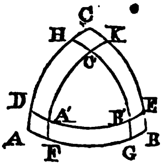
Corol. 3. Vertical spherical angles, such as QPW, QPS, are equal, for each of them is the angle formed by the planes QPS, WPR.

Also, when two arcs cut each other, the two adjacent angles QPW, QPR, when taken together, are always equal to two right angles.

PROPOSITION V.

If from the angular points of a spherical triangle considered as poles three arcs be described forming another triangle, then, reciprocally, the angular points of this last triangle will be the poles of the sides opposite to them in the first.

Let ABC be a spherical triangle.



From the points A, B, C, considered as poles, describe the arcs $B'C'$, $A'C'$, $A'B'$, forming the spherical triangle $A'B'C'$.

Then A' will be the pole of BC, B' of AC, and C' of AB.

For, since B is the pole of $A'C'$, the distance from B to A' is a quadrant.

And, since C is the pole of $A'B'$, the distance from C to A' is a quadrant.

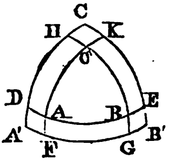
Thus it appears that the point A' is distant by a quadrant from the points B and C.

$\therefore$ A' is the pole of the arc BC.

Similarly, it may be shown that B' is the pole of AC, and C' the pole of AB.

PROPOSITION VI.

The same things being given as in the last proposition, each angle in either of the triangles will be measured by the supplement of the side opposite to it in the other triangle.



Produce the sides of the first triangle to D, E, F, G, H, K.

Then, since A is the pole of $B'C'$, the angle A is measured by the arc EK.

For the same reason, the angles B and C are measured by the arcs DH and FG respectively.

Because B' is the pole of FK, the arc $B'K$ is a quadrant.

Because C' is the pole of DE, the arc $C'E$ is a quadrant.

$$\therefore B'K + C'E = 180^\circ;\ \text{or,}$$

$$B'C' + EK = 180^\circ;$$

$$\therefore EK = 180^\circ - B'C'.$$

Similarly, $$DH = 180^\circ - A'C';$$

$$FG = 180^\circ - A'B'.$$

But the arcs EK, DH, FG are the measures of the angles A, B, C respectively; $\therefore$ $180^\circ - B'C'$, $180^\circ - A'C'$, $180^\circ - A'B'$, or the supplements of $B'C'$, $A'C'$, and $A'B'$ are the measures of these angles.

Again, since A' is the pole of HG, the angle A' is measured by GH.

For the same reason, the angles B', C', are measured by the arcs FK and DE respectively.

Because B is the pole of A′C, the arc BH is a quadrant. Because C is the pole of A′B′, the arc CG is a quadrant.

$$\therefore BH+CG=180^\circ; \text{ or,}$$

$$GH+BC=180^\circ;$$

$$\therefore GH=180^\circ-BC.$$

Similarly,

$$FK=180^\circ-AC;$$

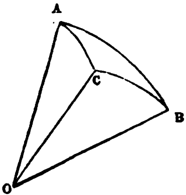
$$DE=180^\circ-AB.$$

And GH, FK, DE are the measures of the angles A′, B′, C′ respectively.

These triangles ABC, A′B′C′, are, from their properties, usually called *polar triangles*, or *supplemental triangles.*

PROPOSITION VII.

In any spherical triangle, any one side is less than the sum of the two others.



Let ABC be a spherical triangle, O the centre of the sphere. Draw the radii OA, OB, OC.

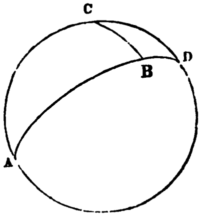
Then the three plane angles AOB, AOC, BOC form a solid angle at the point O, and these three angles are measured by the arcs AB, AC, BC.

But each of the plane angles which form the solid angle is less than the sum of the two others.

Hence each of the arcs AB, AC, BC, which measure these angles, is less than the sum of the two others.

PROPOSITION VIII.

The sum of the three sides of a spherical triangle is less than the circumference of a great circle.



Let ABC be any spherical triangle.

Produce the sides AB, AC, to meet in D.

Then, since two great circles always bisect each other (Prop. 1, Cor.), the arcs ABD, ACD, are semicircles.

Now, in the triangle BCD,

$BC<BD+DC$, by Prop. 7;

$$\therefore AB+AC+BC<AB+BD+AC+DC<ABD+ACD,$$
$$<\text{circumference of great circle.}$$

PROPOSITION IX.

The sum of all the angles in any spherical triangle is less than six right angles, and greater than two right angles.

For, in the first place, every angle of a spherical triangle is less than two right angles; therefore the sum of all the three is less than six right angles. Again, the measure of each angle in the spherical triangle is equal to the semicircumference *minus* the corresponding side of the polar triangle (theor. 6); hence the sum of all three is measured by three semicircumferences *minus* the sum of all the sides of the polar triangle. Now (Prop. 8) this latter sum is less than a circumference; therefore, taking it away from the three semicircumferences, the remainder will be greater than *one* semicircumference, which is the measure of two right angles. Hence the sum of all the angles in a spherical triangle is greater than two right angles.

Corol. 1. The sum of the three angles of a spherical triangle is not constant, like that of the three angles of a plane triangle, but it varies between two right angles and six right angles, without ever arriving at either of these limits. *Two angles, therefore, being given, do not serve to determine the third.*

Corol. 2. A spherical triangle may have two or even three angles right, two or three obtuse. When it has three right angles, it is called a *trirectangular triangle*; the surface of a sphere, therefore, contains *eight* trirectangular triangles.

ELEMENTS OF PLANE TRIGONOMETRY.

CHAPTER I.

1. TRIGONOMETRY was originally considered to be the doctrine of triangles, but in its present improved state it has a much more extensive signification.

2. In estimating angular measures, the right angle is supposed to be the primary one, and to be divided into 90 equal parts, called degrees, marked (°); each degree is divided into 60 equal parts, called minutes, marked (′); each minute is divided into 60 equal parts, called seconds, marked (″); and so on to (‴) thirds, &c. In this division, *one degree* is considered as the *angular unit.*

3. Modern French writers, instead of using this sexagesimal division, use the centesimal.

To reduce French grades into English degrees, or *vice versa*, we use the following formula:

Let E represent the No. of English degrees, and
F " " French grades: then

$$\frac{E}{90}=\frac{F}{100}, \text{ or } \frac{E}{9}=\frac{F}{10};$$

$$\therefore E=\frac{9}{10}F, \text{ or } F=\frac{10}{9}E;$$

$$E=F-\tfrac{1}{10}F, \text{ or } F=E+\tfrac{1}{9}E.$$

Ex. 1. Find the number of degrees and minutes in 46g., 56′ 36″.

$$\begin{array}{rr}
\text{Here } F= & 46.5636 \\
\tfrac{1}{10}F= & \underline{\ \ 4.65636} \\
F-\tfrac{1}{10}F= & 41.90724 \\
 & \underline{60} \\
 & 54.43440 \\
 & \underline{60} \\
 & 26.064
\end{array}$$

$$\therefore E^{\circ}=41^{\circ}\ 54'\ 26''=F-\tfrac{1}{10}F$$

Ex. 2. Find the number of grades, minutes, &c., in 22° 27′ 39″.

$$\begin{array}{r} 60\,|\,\underline{39} \\ 60\,|\,\underline{27.65} \\ E=22.4608+ \\ \frac{1}{9}E=2.4956+ \\ \hline E+\frac{1}{9}E=F=24g.\ 95'\ 64''. \end{array}$$

4. The circumference of a circle is 3.14159+ times its diameter; or, in other words, the ratio of the circumference to the diameter is 3.14159+. Mathematicians represent this number by the Greek letter π.*

$\therefore$ circumference $=\pi D$; where D is the diameter; or
$=2\pi r$, where r is the radius of the circle.

Hence the length of the arc of a quadrant is $\frac{1}{2}\pi r$; of a semicircle, or 180°, it is πr; and of 270°, or three quadrants, it is $\frac{3}{2}\pi r$. Now if any arc, a, subtend an angle of A°, then, since $\frac{1}{2}\pi r$ subtends 90°, and angles are proportional to the arcs which subtend them,

$$A^\circ : 90 :: a : \tfrac{1}{2}\pi r;$$

$$\therefore A^\circ=\frac{180^\circ}{\pi}\cdot\frac{a}{r}\ .\ \ .\ \ .\ \ .\ \ (1)$$

From this expression, any one of the quantities may be found when the others are given.

$$\frac{180^\circ}{\pi}=\frac{180^\circ}{3.14159}=57.29578.$$

Ex. 3. The earth being supposed a sphere, of which the diameter is 7980 miles, find the length of an arc of 1°.

$$A^\circ=57.29578\times\frac{a}{r},\text{ or}$$

$$1^\circ=57.29578\times\frac{a}{3990};$$

$$\therefore a=\frac{3990}{57.29578}=69.6\text{ miles.}$$

Ex. 4. Find the diameter of a globe when an arc of 25° of the meridian measures 4 feet. *Ans.* 18.3346.

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* π is also taken to represent 180°, and $\frac{1}{2}\pi$ for 90°.

NOTE.—When we take 57°.29578 as the unit, $\frac{a}{r}$ is called the circular measure of the angle.

Ex. 5. Find the number of degrees in a circular arc 30 feet in length, whose radius is 25 feet.

Ans. $68^\circ\ 45'\ 17''\ 44'''$.

Ex. 6. Find the number of degrees in an angle of which the arc, divided by the radius, is .7854, the value of π being 3.1416.

$$A^\circ = \frac{180^\circ}{\pi} \times \frac{a}{r} = \frac{180^\circ}{3.1416} \times .7854 = \frac{180^\circ}{4} = 45^\circ.$$

Ex. 7. The interior angles of a rectilineal figure are in arithmetical progression; the least angle is 120°, and the common difference 5°; required the number of sides. (Geom., Book I., Prop. 19.) The sum of all the angles $=2$ right angles taken $(n-2)$ times; that is $(n-2)\pi$, or $180^\circ n-360$. But $120^\circ+125^\circ+130^\circ+$, &c., to n terms$=$sum of all the angles; also (Alg., art. 128). $\left\{240+(n-1)5\right\}\frac{n}{2}=$sum of all the angles.

Equating these, we have

$$\left\{240+(n-1)5\right\}\frac{n}{2}=180n-360;$$
$$(240+5n-5)n=360n-720;$$
$$240n+5n^2-5n=360n-720;$$
$$\therefore n=16 \text{ or } 9.$$

The last is the congruent value of n, since no angle can be so great as 180°; $\therefore$ the figure has 9 sides.

Ex. 8. One regular polygon has two sides more than another, and each of its angles exceeds each angle of the other polygon by 15°; required the number of sides in each.

Let $A=$number of degrees in each angle of the 1st polygon,
$A-15=$ " " " 2d "
$n=$number of sides in the 1st,
$n-2=$ " " 2d;
then $nA=(n-2)\pi$ . . . . (1)
$(n-2)(A-15)=(n-4)\pi$ . . . . (2)

Ans. An octagon or a hexagon.

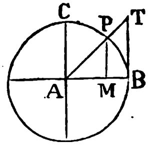
Ex. 9. The angles in one regular polygon are twice as many as in another polygon; and an angle of the former is to an angle of the latter as 3 to 2; to find the number of sides.

Let A = the number of degrees in the angle of the 2d pol.
B = " " " 1st "
n = number of sides of the 2d;
$2n$ = " " " 1st;
$\therefore nA=(n-2)\pi$ . . . . (1)
$2nB=(2n-2)\pi$ . . . . (2)
$A:B::2:3$ . . . . (3)
Ans. $n=4$, and $2n=8$.

(Art. 1.) *Definition of the most important Trigonometrical Terms.*

1. The complement of an angle is what it wants of being 90°. Thus, if A be any angle, the complement of A is $(90°-A)$.

2. The supplement of an angle is what it wants of being equal to 180°, or to two right angles. Therefore, if A be any angle, its supplement will be $(180°-A)$.



3. In the annexed diagram, the ratio of PM to the radius of the circle is called the sine of the angle PAB, and AM to the radius is the cosine; or, $\frac{PM}{AB}=\sin. A$, and $\frac{AM}{AB}=\cos. A$.

4. The ratio of BT to the radius of the circle is called the tangent of the angle PAB; or, $\frac{BT}{AB}=\tan. A$.

5. The ratio of AT to the radius of the circle is called the secant of the angle PAB; or, $\frac{AT}{AB}=\sec. A$.

6. The ratio of BM to the radius of the circle is called the versed sine of the angle PAB; or, $\frac{BM}{AB}=\text{vers. sin. } A$.

7. The sine of the complement of any angle is called the cosine of that angle; or,

$$\sin. (90°-A)=\cos. A.$$
$$\cos. (90°-A)=\sin. A.$$

8. The tangent of the complement of any given angle is called the cotangent of that angle; or,

$$\tan. (90°-A)=\cot. A.$$
$$\cot. (90°-A)=\tan. A.$$

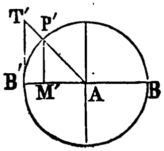
9. The secant of the complement of any angle is called cosecant of that angle; or,

$$\text{sec. } (90^\circ - A) = \text{cosec. } A.$$
$$\text{cosec. } (90^\circ - A) = \text{sec. } A.$$

10. The versed sine of the complement of any angle is called the coversed sine of that angle; or,

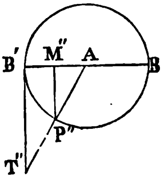
$$\text{vers. sin. } (90^\circ - A) = \text{covers. sin. } A.$$
$$\text{covers. sin. } (90^\circ - A) = \text{vers. sin. } A.$$

11. The foregoing definitions relate to an angle less than a right angle; but they are equally applicable to any angle, whatever may be its magnitude. For, let P′AB be greater than a right angle and less than two right angles; then



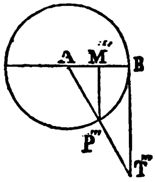
$$\frac{P'M'}{AB} = \text{sin. } P'AB;$$
$$\frac{AM'}{AB} = \text{cos. } P'AB;$$
$$\frac{B'T'}{AB} = \text{tan. } P'AB;$$
$$\frac{AT'}{AB} = \text{sec. } P'AB;$$
$$\frac{B'M}{AB} = \text{ver. sin. } P'AB.$$

12. Again: let the angle be situate in the third quadrant —that is, greater than two and less than three right angles —then we shall have



$$\frac{P''M''}{AB} = \text{sin. } P''AB;$$
$$\frac{AM''}{AB} = \text{cos. } P'''AB;$$
$$\frac{B'T''}{AB} = \text{tan. } P''AB;$$
$$\frac{AT''}{AB} = \text{sec. } P''AB;$$
$$\frac{B'M''}{AB} = \text{vers. sin. } P''AB.$$

13. Finally, let the angle be placed in the fourth quadrant; then



$$\frac{P'''M'''}{AB}=\text{sin. }P'''AB;$$

$$\frac{AM'''}{AB}=\text{cos. }P'''AB;$$

$$\frac{BT'''}{AB}=\text{tan. }P'''AB;$$

$$\frac{AT'''}{AB}=\text{sec. }P'''AB;$$

$$\frac{BM'''}{AB}=\text{vers. sin. }P'''AB;$$

14. The *sine of an arc* is the perpendicular let fall from one extremity of the arc on the diameter passing through the other extremity. Thus, MP is the sine of the arc BP; in like manner, AM is the *cosine;* BM is the *versine;* BT, the *tangent*; AT, the *secant*, &c., of the arc BP. Hence, when the radius is unity, the sine, &c., of the arc become the sine, &c., of the angle.

(Art. 2.) *General Relations between Trigonometrical Quantities.*

1. Since AMP is a right-angled triangle, and AP the hypothenuse, $PM^2+AM^2=AB^2$.

Divide by AB^2, and we have

$$\frac{PM^2}{AB^2}+\frac{AM^2}{AB^2}=1; \text{ that is,}$$

$$\text{sin.}^2 A+\text{cos.}^2 A=1 \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \text{sin. } A=\sqrt{1-\text{cos.}^2 A}; \text{ or cos. } A=\sqrt{(1-\text{sin.}^2 A)}.$$

The triangles PMA, TBA, are similar; therefore

$$\frac{PM}{AM}=\frac{TB}{AB}; \text{ but } \frac{PM}{AM}=\frac{PM}{AB}\div\frac{AM}{AB}=\frac{\text{sin. }A}{\text{cos. }A};$$

hence, $$\frac{\text{sin. }A}{\text{cos. }A}=\text{tan. }A \quad . \quad . \quad . \quad . \quad (2)$$

In equation (2), instead of A, substitute $(90°-A)$;

then $$\frac{\text{sin. }(90°-A)}{\text{cos. }(90°-A)}=\text{tan. }(90°-A);$$

or, $$\frac{\text{cos. }A}{\text{sin. }A}=\text{cot. }A \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Multiplying equations (2) and (3) together, we have

$$\text{tan. }A\,.\,\text{cot. }A=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Again: $\frac{\text{AT}}{\text{AB}}=\frac{\text{AP}}{\text{AM}}=1\div\frac{\text{AM}}{\text{AP}}$; or,

$$\text{sec. A}=\frac{1}{\text{cos. A}},$$

$$\text{sec. A . cos. A}=1;\ \therefore \text{sec. A}=\frac{1}{\text{cos. A}}. \quad . \quad (5)$$

But $\text{cosec. A}=\text{sec. }(90^\circ-\text{A}) \therefore \text{cosec. A}=\frac{1}{\text{cos. }(90^\circ-\text{A})}=$ $\frac{1}{\text{sin. A}}$; or, cosec. A . sin. A = 1. . . . . . . (6)

$$\text{AB}^2+\text{BT}^2=\text{AT}^2.\ \text{Divide by AB}^2\text{, then}$$

$$1+\frac{\text{BT}^2}{\text{AB}^2}=\frac{\text{AT}^2}{\text{AB}^2};\ \text{that is,}$$

$$1+\text{tan.}^2\text{A}=\text{sec.}^2\text{A} \quad . \ . \ . \ . \ . \ . \quad (7)$$

By squaring equation (3), and adding unity to each member, we have

$$1+\frac{\text{cos.}^2\text{A}}{\text{sin.}^2\text{A}}=1+\text{cot.}^2\text{A} \therefore 1+\text{cot.}^2\text{A}=\frac{\text{sin.}^2\text{A}+\text{cos.}^2\text{A}}{\text{sin.}^2\text{A}};$$

but, $\text{sin.}^2\text{A}+\text{cos.}^2\text{A}=1 \therefore 1+\text{cot.}^2\text{A}=\frac{1}{\text{sin.}^2\text{A}}$; hence

$$1+\text{cot.}^2\text{A}=\text{cosec.}^2\text{A} \quad . \ . \ . \ . \ . \ . \ . \quad (8)$$

$$\frac{\text{MB}}{\text{AB}}=\frac{\text{AB}-\text{AM}}{\text{AB}}=1-\frac{\text{AM}}{\text{AB}};\ \text{hence}$$

$$\text{versin. A}=1-\text{cos. A} \quad . \ . \ . \ . \quad (9)$$

$$\text{coversin. A}=\text{versin. }(90^\circ-\text{A})=1-\text{cos. }(90^\circ-\text{A});$$

$$\therefore \text{coversin. A}=1-\text{sin. A} \quad . \quad (10)$$

11. Trigonometrical quantities are affected by the signs *plus* and *minus.*

When the angle A = 0, sin. A = 0. While A varies from 0° to 90°, sin. A increases from 0 to 1; when A = 90°, sin. A = 1.

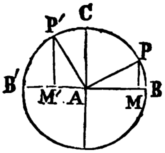
While A increases from 90° to 180°, sin. A decreases from 1 to 0; when A = 180°, sin. A = 0.

While A increases from 180° to 270°, sin. A varies from 0 to −1; when A = 270°, sin. A = −1.

While A increases from 270° to 360°, sin. A varies from 1 to 0; when A = 360°, sin. A = 0; *and so on for successive revolutions.*

The cosine varies in a similar manner, being *plus* in the first and fourth quadrants, and *minus* in the second and third. The variation in the magnitude of the sines and cosines being known, the other trigonometrical quantities may be easily determined.

(Art. 3.) *Relations existing between the trigonometrical functions of Angles less than* 90°, *and those that are greater than* 90°.



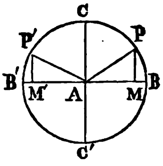
Draw AP, making any angle A; let fall PM perpendicular on AB; draw AP′, making with AC the angle CAP′=PAB =A; and from P′ let fall P′M′ perpendicular on AB′; then the angle P′AB=90° +A. The two triangles PAM, P′AM′, have the side PA of the one equal to P′A of the other; also the angles at M and M′ right angles, and the angle APM of the one equal to P′AM′ of the other; therefore the two triangles are equal in every respect, and PM=AM′, and AM=P′M′. Divide this last equation by AB, and we have

$$\frac{AM}{AB}=\frac{P'M'}{AB}\text{; or cos. PAB=sin. P}'\text{AB; that is,}$$

$$\text{cos. A=sin. }(90°+\text{A}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Again: $\frac{AM'}{AB}=\frac{PM}{AB}$; or $-$cos. P′AB=sin. PAB;

$$\therefore \text{cos. }(90°+\text{A})=-\text{sin. A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$



But, instead of AP′ making an angle with AC, let it make that angle with AB; then the angle P′AB is equal to 180°$-$A. The two triangles PAB, P′AB, are manifestly equal in all respects, and PM=P′M′, AM=AM′. Divide the first of these equations by AB, we have

$$\frac{PM}{AB}=\frac{P'M'}{AB}\text{; or sin. A=sin. }(180°-\text{A}) \quad . \quad (13)$$

Hence *the sine of an angle is equal to the sine of its supplement.*

Again: dividing the other equation by AB, we have

$$\frac{AM}{AB}=\frac{AM'}{AB}\text{; or cos. A=}-\text{cos. }(180°-\text{A}) \quad (14)$$

That is, *the cosine of an angle and the cosine of its supplement are equal in absolute magnitude, but have opposite signs.*

If we draw AP′, making with AB′ an angle equal to A in the third quadrant, we shall find, in like manner,

$$\sin.(180^\circ+A)=-\sin. A \quad . \quad . \quad . \quad (15)$$

$$\cos.(180^\circ+A)=\cos.(180^\circ-A) \quad . \quad (16)$$

$$=-\cos. A \quad . \quad . \quad (17)$$

If we draw AP′, making with AC′ an angle C′AP=A, then

$$\sin.(270^\circ-A)=-\cos. A \quad . \quad . \quad (18)$$

$$\cos.(270^\circ-A)=-\sin. A \quad . \quad . \quad (19)$$

In like manner we may prove for angles in the fourth quadrant. These relations between the sines and cosines being established, the corresponding relations between the other trigonometrical functions may be deduced immediately from the preceding equations. Thus,

$$\tan.(90^\circ+A)=\frac{\sin.(90^\circ+A)}{\cos.(90^\circ+A)}=\frac{\cos. A}{-\sin. A}=-\cot. A \quad (20)$$

$$\tan.(180^\circ-A)=\frac{\sin.(180^\circ-A)}{\cos.(180^\circ-A)}=\frac{\sin. A}{-\cos. A}=-\tan. A \quad (21)$$

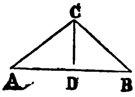
And so for all the rest.

Although the preceding investigations have been confined to angles not greater than 360°, still the same principles may be applied without any difficulty to angles of any magnitude whatever.

Having the sines given, we can calculate the table of cosines, tangents, cotangents, secants, and cosecants by the foregoing equations.

(ART. 4.) *To find the sine, cosine, &c., of* 30°, 45°, *and* 60°.

Let the angle be 60°. In the equilateral triangle ACB, let fall the perpendicular CD, which bisects both the angle ACB and the base AB. (Geom., B. I, Prop. 3.)



Now, $\frac{AD}{AC}=\sin. ACD=\sin. 30^\circ$; but $AD=\frac{1}{2}AC$;

$$\therefore \sin. 30^\circ=\frac{\frac{1}{2}AC}{AC}=\tfrac{1}{2}.$$

$\cos. 30^\circ=\sqrt{1-\sin.^2 30^\circ}=\sqrt{(1-\frac{1}{4})}=\sqrt{\frac{3}{4}}=\frac{1}{2}\sqrt{3}$; but

$$\cos. 30^\circ = \sin. 60^\circ \therefore \sin. 60^\circ = \tfrac{1}{2}\sqrt{3};$$
$$\cos. 60^\circ = \sin. 30^\circ \therefore \cos. 60^\circ = \tfrac{1}{2};$$
$$\therefore \tan. 30^\circ = \frac{\sin. 30^\circ}{\cos. 30^\circ} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}};$$
$$\cot. 30^\circ = \frac{\cos. 30^\circ}{\sin. 30^\circ} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3};$$
$$\sec. 30^\circ = \frac{1}{\cos. 30^\circ} = \frac{1}{\frac{1}{2}\sqrt{3}} = \frac{2}{\sqrt{3}};$$
$$\operatorname{cosec}. 30^\circ = \frac{1}{\sin. 30^\circ} = \frac{1}{\frac{1}{2}} = 2.$$

In like manner, find *tan.*, *cot.*, *sec.*, and *cosec.* of 60°.

When the angle is 45°,

$$\sin. 45^\circ = \cos. (90^\circ - 45^\circ) = \cos. 45^\circ.$$

Now, since $\cos.^2 45^\circ + \sin.^2 45^\circ = 1$, and $\cos. 45^\circ = \sin. 45^\circ$,

we have $$2 \cos.^2 45^\circ = 1 \therefore \cos.^2 45^\circ = \tfrac{1}{2};$$

or $$\cos. 45^\circ = \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2};$$
$$\therefore \sin. 45^\circ = \tfrac{1}{2}\sqrt{2};$$

hence $$\tan. 45^\circ = \frac{\sin. 45^\circ}{\cos. 45^\circ} = \frac{\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}} = 1;$$
$$\cot. 45^\circ = \frac{\cos. 45^\circ}{\sin. 45^\circ} = 1;$$
$$\sec. 45^\circ = \frac{1}{\cos. 45^\circ} = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2};$$
$$\operatorname{cosec}. 45^\circ = \frac{1}{\sin. 45^\circ} = \sqrt{2}.$$

Ex. 10. If $\tan. x = \frac{4}{3}$, find the *sin.* and *versin.* and *cos.* x.

$$\tan. x = \frac{\sin. x}{\cos. x} \therefore \frac{\sin. x}{\cos. x} = \tfrac{4}{3};$$
$$3 \sin. x = 4 \cos. x;$$
$$= 4\sqrt{(1 - \sin.^2 x)};$$
$$9 \sin.^2 x = 16 - 16 \sin.^2 x;$$
$$25 \sin.^2 x = 16 \therefore \sin. x = \tfrac{4}{5};$$
$$\cos. x = \tfrac{3}{5}, \quad \text{vers. } x = \tfrac{2}{5}.$$

Ex. 11. Given $6 \sin.^2 x = 5 \cos. x$, to find sin., cos., and tan. x.

Ans. $\sin. x = \frac{1}{3}\sqrt{5}$;
$\cos. x = \frac{2}{3}$ or $-\frac{3}{2}$;
$\tan. x = \frac{1}{2}\sqrt{5}$.

Ex. 12. Given $\sin. x \cos. x = \frac{1}{4}\sqrt{3}$, to find $\sin. x$ and $\cos. x$.

Ans. $\sin. x = \frac{1}{2}\sqrt{3}$ or $\frac{1}{2}$;
$\cos. x = \frac{1}{2}$ or $\frac{1}{2}\sqrt{3}$.

Ex. 13. Given $\tan. x + \cot. x = \frac{4}{\sqrt{3}}$, to find $\tan. x$ and $\cot. x$.

Ans. $\tan. x = \sqrt{3}$ or $\frac{1}{\sqrt{3}}$;
$\cot. x = \frac{1}{\sqrt{3}}$ or $\sqrt{3}$.

Ex. 14. Given $3 \sin. x + 5\sqrt{3} . \cos. x = 9$, to find $\sin. x$.

Ans. $\sin. x = \frac{1}{2}$ or $\frac{1}{7}$.

Ex. 15. Given $\sin. x + \sin. y = \frac{1+\sqrt{3}}{2}$

$\sin. x \sin. y = \frac{1}{4}\sqrt{3}$, to find $\sin. x$ and $\sin. y$.

Ans. $\sin. x = \frac{1}{2}$;
$\sin. y = \frac{1}{2}\sqrt{3}$.

Ex. 16. Prove that $\sec.^2 x \operatorname{cosec.}^2 x = \sec.^2 x + \operatorname{cosec.}^2 x$.

Ex. 17. Given $\tan.^2 x + 4 \sin.^2 x = 6$, to find $\sin. x$.

Ans. $\sin. x = \frac{1}{2}\sqrt{3}$.

Ex. 18. Given $\sin. x(\sin. x - \cos. x) = \frac{4}{25}$, to find $\sin. x$.

Ans. $\sin. x = \frac{4}{5}$.

Ex. 19. Given $6 \tan. x + 12 \cot. x = 5\sqrt{3} . \sec. x$, to find $\tan x$.

Ans. $\tan. x = \sqrt{3}$.

Ex. 20. Given $\tan. x + \cot. x = 2$, to find $\tan x$.

Ans. $\tan. x = 1$.

Ex. 21. Given $\tan.^3 x + 7 \tan. x = 22$, to find $\tan. x$.

Multiply each member of the equation by $\tan. x$, and we have
$$\tan.^4 x + 7 \tan.^2 x = 22 \tan. x;$$
add $4 \tan.^2 x$ to both members, and
$$\tan.^4 x + 11 \tan.^2 x = 4 \tan.^3 x + 22 \tan. x;$$
comp. □, and extract the root, and reduce, we have
$$\tan. x = 2.$$

Ex. 22. Given $\tan.^2 x + \frac{1}{8} = \frac{1}{\tan. x} + \frac{1}{8} \cot. x$, to find $\tan. x$.
$$\tan.^3 x + \tfrac{1}{8} \tan. x = 1 + \tfrac{1}{8} = \tfrac{9}{8};$$
multiply by $\tan. x$, and we have
$$\tan.^4 x + \tfrac{1}{8} \tan.^2 x = \tfrac{9}{8} \tan. x.$$
If we now add $\tan.^2 x$ to both members, we can complete the square and extract the root. *Ans.* $\tan. x = 1$.

Ex. 23. Given $4 \sec.^2 x = 195 \cot. x - 117$, to find $\tan. x$.

Ans. $\tan. x = \frac{3}{2}$.

Ex. 24. Given $\sin.^4 x - 2\sin.^2 x - 1 = 2\sin. x - \cos.^2 x$, to find $\sin. x$. *Ans.* $\sin. x = -1$.

CHAPTER II.

PROPOSITION I. THEOREM I.

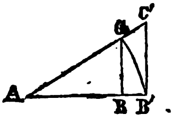
(Art. 5.) 1. *In any right-angled triangle, the ratio which the side opposite to one of the acute angles bears to the hypothenuse is the sine of that angle; 2. the ratio which the side adjacent to one of the acute angles bears to the hypothenuse is the cosine of that angle; 3. and the ratio which the side opposite to one of the acute angles bears to the side adjacent to that angle is the tangent of that angle.*

Let ABC be any plane triangle right-angled at B;

then $\frac{CB}{AC} = \sin. A$; or $\frac{AB}{AC} = \sin. C$. (1)

$\frac{AB}{AC} = \cos. A$; or $\frac{BC}{AC} = \cos. C$. (2)

$\frac{BC}{AB} = \tan. A$; or $\frac{AB}{BC} = \tan. C$. (3)



Proof.—From A as a centre, with radius AC, describe an arc of a circle; produce AB to meet the circumference in B′; from B′ draw B′C′, a tangent to the arc at B′; produce AC to meet B′C′ in C′; then, from definitions (3) and (4),

$$\frac{BC}{AB'} = \sin. A,\ \frac{AB}{AB'} = \cos. A, \text{ and } \frac{B'C'}{AB'} = \tan. A.$$

But $AB' = AC$, whence $\frac{BC}{AC} = \sin. A$; $\frac{AB}{AC} = \cos. A$.

And the triangles ABC, AB′C′, being similar,

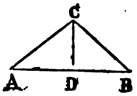
$$\frac{B'C'}{AC} = \frac{BC}{AB} \therefore \frac{BC}{AB} = \tan. A.$$

PROPOSITION II. THEOREM II.

In any plane triangle, the ratio of any two of the sides is equal to the ratio of the sines of the angles opposite to them.

Let ABC be a plane triangle; it is required to prove that

$$\frac{CB}{CA}=\frac{\sin. A}{\sin. B};\ \frac{CB}{BA}=\frac{\sin. A}{\sin. C};$$
$$\frac{CA}{BA}=\frac{\sin. B}{\sin. C}.$$



Draw CD perpendicular to AB; then, by theorem 1, $\frac{DC}{CA}=\sin. A$; $\frac{DC}{CB}=\sin. B$. Divide the first of these equations by the second, we have $\frac{DC}{CA}\div\frac{DC}{CB}=\frac{\sin. A}{\sin. B}$, or $\frac{CB}{CA}=\frac{\sin. A}{\sin. B}$.

In like manner, by letting fall perpendiculars from B and A upon the sides AC, CB, we can prove that

$$\frac{CB}{BA}=\frac{\sin. A}{\sin. C};\ \frac{CA}{BA}=\frac{\sin. B}{\sin. C}.$$

In treating of plane triangles, we make use of the capital letters A, B, C to designate the three angles, and the corresponding small letters *a*, *b*, *c* to represent the sides opposite to them. According to this notation, this last theorem will be

$$\frac{a}{b}=\frac{\sin. A}{\sin. B};\ \frac{a}{c}=\frac{\sin. A}{\sin. C};\ \frac{b}{c}=\frac{\sin. B}{\sin. C}.$$

(ART. 6.) *Investigation of General Formulas.*

PROBLEM I.

Given the sines and cosines of two angles, to find the sine of their sum.

Let ABC be a plane triangle (see the last figure); then, by equation 2, theorem 1,

$$BD=BC\cos. B,\ AD=AC\cos. A;\ \text{but}$$
$$AB=BD+AD\qquad =BC\cos. B+AC\cos. A.$$

Divide each term of this equation by AB, we obtain

$$1=\frac{BC}{AB}\cos. B+\frac{AC}{AB}\cos. A;\ \text{but by theorem 2,}$$

$\frac{C}{AB}=\frac{\sin. A}{\sin. C}$, and $\frac{AC}{AB}=\frac{\sin. B}{\sin. C}$. Substitute these in the above, we have $1=\frac{\sin. A\cos. B}{\sin. C}+\frac{\sin. B\cos. A}{\sin. C}$;

$$\therefore \sin. C=\sin. A\cos. B+\sin. B\cos. A;$$

but, since ABC is a plane triangle, $A+B+C=180^\circ \therefore C=180^\circ-(A+B)$; and $\sin. C=\sin. \{180^\circ-(A+B)\}=\sin. (A+B)$, because $180^\circ-(A+B)$ is the supplement of $(A+B)$.

$$\therefore \sin. (A+B)=\sin. A \cos. B+\sin. B \cos. A \quad . \quad . \quad (1)$$

Equation (1), from its great importance, is called the *fundamental formula of Plane Trigonometry*, and nearly the whole science may be derived from it.

PROBLEM II.

To find the sine of the difference of two angles, the sines and cosines of the two angles being given.

By equation (1), $\sin. (A+B)=\sin. A \cos. B+\sin. B \cos. A$; for A substitute $(180^\circ-A)$, the above will become

$$\sin. \{(180^\circ-A)+B\}=\sin. \{180^\circ-(A-B)\}=\sin. (180^\circ-A) \cos. B+\sin. B \cos. (180^\circ-A).$$

But $\sin. \{180^\circ-(A-B)\}=\sin. (A-B)$,
and $\sin. (180^\circ-A)=\sin. A$; $\cos. (180^\circ-A)=-\cos. A$.
Substitute these in the above expression, it becomes

$$\sin. (A-B)=\sin. A \cos. B-\sin. B \cos. A \quad . \quad . \quad (2)$$

PROBLEM III.

Given the sines and cosines of two angles, to find the cosine of their sum.

Formula (1), $\sin. (A+B)=\sin. A \cos. B+\sin. B \cos. A$; for A substitute $(90^\circ+A)$, we have

$$\sin. \{(90^\circ+A)+B\}=\sin. (90^\circ+A) \cos. B+\sin. B \cos. (90^\circ+A).$$

But $\sin. \{90^\circ+(A+B)\}=\cos. (A+B)$, and
$\sin. (90^\circ+A)=\cos. A$, and $\cos. (90^\circ+A)=-\sin. A$.
Substituting, therefore, these values in the above expression, it becomes

$$\cos. (A+B)=\cos. A \cos. B-\sin. A \sin. B \quad . \quad (3)$$

PROBLEM IV.

To find the cosine of the difference of two angles.

Formula (1), $\sin. (A+B)=\sin. A \cos. B+\sin. B \cos. A$; for A substitute $(90^\circ-A)$, it will become

$$\sin. \{(90^\circ-A)+B\}=\sin. \{90^\circ-(A-B)\}=\sin. (90^\circ-A) \cos. B+\sin. B \cos. (90^\circ-A).$$

But $\sin.\{90^\circ-(A-B)\}=\cos.(A-B)$;

$$\sin.(90^\circ-A)=\cos.A;$$

$$\cos.(90^\circ-A)=\sin.A.$$

Substitute, therefore, these values in the above expression, it becomes

$$\cos.(A-B)=\cos.A\cos.B+\sin.A\sin.B \quad . \quad . \quad (4)$$

PROBLEM V.

To find the tangent of the sum of two angles.

We have by Chap. I., equation 2,

$$\tan.(A+B)=\frac{\sin.(A+B)}{\cos.(A+B)}=\frac{\sin.A\cos.B+\sin.B\cos.A}{\cos.A\cos.B-\sin.A\sin.B}.$$

Divide both the numerator and denominator of this fraction by cos. A cos. B, we shall have

$$\tan.(A+B)=\frac{\frac{\sin.A\cos.B}{\cos.A\cos.B}+\frac{\sin.B\cos.A}{\cos.A\cos.B}}{1-\frac{\sin.A\sin.B}{\cos.A\cos.B}}.$$

Simplifying,

$$\tan.(A+B)=\frac{\tan.A+\tan.B}{1-\tan.A\tan.B} \quad . \quad . \quad . \quad . \quad (5)$$

PROBLEM VI.

Given the tangents of two angles, to find the tangent of their difference.

(Chapter I., equation 2):

$$\tan.(A-B)=\frac{\sin.(A-B)}{\cos.(A-B}=\frac{\sin.A\cos.B-\sin.B\cos.A}{\cos.A\cos.B+\sin.A\sin.B}.$$

Dividing both the numerator and denominator by cos. A cos. B, it becomes

$$\tan.(A-B)=\frac{\frac{\sin.A\cos.B}{\cos.A\cos.B}-\frac{\sin.B\cos.A}{\cos.A\cos.B}}{1+\frac{\sin.A\sin.B}{\cos.A\cos.B}}.$$

Simplifying, we have

$$\tan.(A-B)=\frac{\tan.A-\tan.B}{1+\tan.A\tan.B} \quad . \quad . \quad . \quad . \quad (6)$$

The sum of the equations (1) and (2) is

$$\sin.(A+B)+\sin.(A-B)=2\sin.A\cos.B \quad . \quad (7)$$

Their difference is

$$\sin.(A+B)-\sin.(A+B)=2\sin.B\cos.A \quad . \quad (8)$$

The sum of (3) and (4) is

$$\cos.(A+B)+\cos.(A-B)=2\cos.A\cos.B \quad . \quad (9)$$

Their difference is

$$\cos.(A+B)-\cos.(A+B)=2\sin.A\sin.B \quad . \quad (10)$$

If, in equations (7), (8), (9), and (10), we make

$A+B=p$, and $A-B=q$, that is,

$A=\frac{1}{2}(p+q) \quad B=\frac{1}{2}(p-q)$, we shall have

$$\sin.p+\sin.q=2\sin.\tfrac{1}{2}(p+q)\cos.\tfrac{1}{2}(p-q) \quad . \quad (11)$$

$$\sin.p-\sin.q=2\sin.\tfrac{1}{2}(p-q)\cos.\tfrac{1}{2}(p+q) \quad . \quad (12)$$

$$\cos.p+\cos.q=2\cos.\tfrac{1}{2}(p+q)\cos.\tfrac{1}{2}(p-q) \quad . \quad (13)$$

$$\cos.q-\cos.p=2\sin.\tfrac{1}{2}(p+q)\sin.\tfrac{1}{2}(p-q) \quad . \quad (14)$$

Divide (11) by (12), we obtain

$$\frac{\sin.p+\sin.q}{\sin.p-\sin.q}=\frac{\sin.\frac{1}{2}(p+q)\cos.\frac{1}{2}(p-q)}{\cos.\frac{1}{2}(p+q)\sin.\frac{1}{2}(p-q)}.$$

Reducing the second member, it becomes

$$\frac{\sin.p+\sin.q}{\sin.p-\sin.q}=\tan.\tfrac{1}{2}(p+q)\cot.\tfrac{1}{2}(p-q)=\frac{\tan.\frac{1}{2}(p+q)}{\tan.\frac{1}{2}(p-q)}. \quad (15)$$

Dividing (14) by (13), we have

$$\frac{\cos.q-\cos.p}{\cos.q+\cos.p}=\tan.\tfrac{1}{2}(p+q)\tan.\tfrac{1}{2}(p-q);$$

$$=\frac{\tan.\frac{1}{2}(p+q)}{\cot.\frac{1}{2}(p-q)}=\frac{\tan.\frac{1}{2}(p-q)}{\cot.\frac{1}{2}(p+q)} \quad . \quad (16)$$

(15) may also be written

$$\frac{\sin.p+\sin.q}{\sin.p-\sin.q}=\frac{\tan.\frac{1}{2}(p+q)}{\tan.\frac{1}{2}(p-q)}=\frac{\cot.\frac{1}{2}(p-q)}{\cot.\frac{1}{2}(p+q)} \quad . \quad . \quad (17)$$

If, in (13) and (14), we make $q=0$, we shall have

$$1+\cos.p=2\cos.^2\tfrac{1}{2}p, \text{ and } 1-\cos.p=2\sin.^2\tfrac{1}{2}p \quad . \quad . \quad (a)$$

$$\frac{\sin.p+\sin.q}{\cos.p+\cos.q}=\frac{\sin.\frac{1}{2}(p+q)}{\cos.\frac{1}{2}(p+q)}=\tan.\tfrac{1}{2}(p+q) \quad . \quad (b)$$

$$\frac{\sin.p+\sin.q}{\cos.q-\cos.p}=\frac{\cos.\frac{1}{2}(p-q)}{\sin.\frac{1}{2}(p-q)}=\cot.\tfrac{1}{2}(p-q) \quad . \quad (c)$$

$$\frac{\sin.p-\sin.q}{\cos.p+\cos.q}=\frac{\sin.\frac{1}{2}(p-q)}{\cos.\frac{1}{2}(p-q)}=\tan.\tfrac{1}{2}(p-q) \quad . \quad (d)$$

$$\frac{\sin.p-\sin.q}{\cos.q-\cos.p}=\frac{\cos.\frac{1}{2}(p+q)}{\sin.\frac{1}{2}(p+q)}=\cot.\tfrac{1}{2}(p+q) \quad . . \quad (e)$$

$$\frac{\cos. p+\cos. q}{\cos. q-\cos. p}=\frac{\cos. \frac{1}{2}(p+q)\cos. \frac{1}{2}(p-q)}{\sin. \frac{1}{2}(p+q)\sin. \frac{1}{2}(p-q)}=\frac{\cot. \frac{1}{2}(p+q)}{\tan. \frac{1}{2}(p-q)} \quad (f)$$

$$\frac{\sin. p+\sin. q}{\sin. (p+q)}=\frac{\sin. \frac{1}{2}(p+q)\cos. \frac{1}{2}(p-q)}{\sin. \frac{1}{2}(p+q)\cos. \frac{1}{2}(p+q)}=\frac{\cos. \frac{1}{2}(p-q)}{\cos. \frac{1}{2}(p+q)} \quad (g)$$

$$\frac{\sin. p-\sin. q}{\sin. (p+q)}=\frac{\sin. \frac{1}{2}(p-q)\cos. \frac{1}{2}(p+q)}{\sin. \frac{1}{2}(p+q)\cos. \frac{1}{2}(p+q)}=\frac{\sin. \frac{1}{2}(p-q)}{\sin. \frac{1}{2}(p+q)} \quad (h)$$

PROBLEM VII.

(ART. 7.) *To determine the sine of twice a given angle.*

Formula 1:

$$\sin. (A+B)=\sin. A \cos. B+\sin. B \cos. A.$$

Let $B=A$, then the equation becomes

$$\sin. 2A=\sin. A \cos. A+\sin. A \cos. A$$
$$=2 \sin. A \cos. A \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Cor. In (18), just obtained, let $\frac{1}{2}A$ be substituted for A; then

$$\sin. A=2 \sin. \tfrac{1}{2}A \cos. \tfrac{1}{2}A \quad . \quad . \quad . \quad . \quad (19)$$

PROBLEM VIII.

To determine the cosine of twice a given angle.

By formula (3):

$$\cos. (A+B)=\cos. A \cos. B-\sin. A \sin. B.$$

Let $B=A$, the above becomes

$$\cos. 2A=\cos.^2 A-\sin.^2 A \quad . \quad . \quad (20)$$

But $\sin.^2 A=1-\cos.^2 A$. Substitute this for $\sin.^2 A$, then

$$\cos. 2A=2 \cos.^2 A-1 \quad . \quad . \quad . \quad (21)$$

Again: since $\cos.^2 A=1-\sin.^2 A$, we have

$$\cos. 2A=1-2 \sin.^2 A \quad . \quad . \quad . \quad (22)$$

PROBLEM IX.

By formula (5), of the present chapter:

$$\tan. (A+B)=\frac{\tan. A+\tan. B}{1-\tan. A \tan. B}.$$

Let $B=A$, the above becomes

$$\tan. 2A=\frac{2 \tan. A}{1-\tan.^2 A} \quad . \quad . \quad . \quad . \quad (23)$$

In like manner deduce the following:

PROBLEM X.

$$\cot. 2A=\frac{\cot.^2 A-1}{2 \cot. A} \quad . \quad . \quad . \quad . \quad (24)$$

PROBLEM XI.

$$\text{sec. } 2A=\frac{\text{sec.}^2 A \text{ cosec.}^2 A}{\text{cosec.}^2 A-\text{sec.}^2 A} \quad . \quad (25)$$

PROBLEM XII.

$$\left.\begin{aligned}\text{cosec. } 2A&=\frac{\text{sec.}^2 A \text{ cosec.}^2 A}{2 \text{ sec. } A \text{ cosec. } A}\\ &=\tfrac{1}{2} \text{ sec. } A \text{ cosec. } A\end{aligned}\right\} \quad \begin{aligned}&. \quad (26)\\ &. \quad (27)\end{aligned}$$

PROBLEM XIII.

To determine the sine of half a given angle.

By Formula (22):

$$\text{cos. } 2A=1-2 \text{ sin.}^2 A.$$

For A substitute $\frac{1}{2}$A, and we have

$$\text{cos. } 2 . (\tfrac{1}{2}A)=1-2 \text{ sin.}^2 \tfrac{1}{2}A;$$

or

$$\text{cos. } A=1-2 \text{ sin.}^2 \tfrac{1}{2}A;$$

$$2 \text{ sin.}^2 \tfrac{1}{2}A=1-\text{cos. } A;$$

$$\text{sin.}^2 \tfrac{1}{2}A=\frac{1-\text{cos. } A}{2};$$

$$\text{sin. } \tfrac{1}{2}A=\sqrt{\frac{1-\text{cos. } A}{2}} \quad . \quad . \quad (28)$$

PROBLEM XIV.

To determine the cosine of half a given angle.

By Formula (21):

$$\text{cos. } 2A=2 \text{ cos.}^2 A-1.$$

For A substitute $\frac{1}{2}$A, we have

$$\text{cos. } A=2 \text{ cos.}^2 \tfrac{1}{2}A-1;$$

or

$$2 \text{ cos.}^2 \tfrac{1}{2}A=1+\text{cos. } A;$$

$$\text{cos. } \tfrac{1}{2}A=\sqrt{\frac{1+\text{cos. } A}{2}} \quad . \quad . \quad . \quad (29)$$

PROBLEM XV.

To determine the tangent of half a given angle.

Divide equation (28) by (29):

$$\frac{\text{sin. } \frac{1}{2}A}{\text{cos. } \frac{1}{2}A}=\sqrt{\frac{1-\text{cos. } A}{1+\text{cos. } A}};$$

or

$$\text{tan. } \tfrac{1}{2}A=\sqrt{\frac{1-\text{cos. } A}{1+\text{cos. } A}} \quad . \quad . \quad (30)$$

Multiply both numerator and denominator of the fraction by $\sqrt{1-\text{cos. A}}$, we have

$$\text{tan. } \tfrac{1}{2}\text{A} = \frac{1-\text{cos. A}}{\text{sin. A.}} \quad . \quad . \quad . \quad . \quad (31)$$

Multiply both numerator and denominator of (30) by $1+\text{cos. A}$, we have

$$\text{tan. } \tfrac{1}{2}\text{A} = \frac{\text{sin. A}}{1+\text{cos. A}} \quad . \quad . \quad . \quad . \quad (32)$$

In like manner deduce

$$\text{cot. } \tfrac{1}{2}\text{A} = \sqrt{\frac{1+\text{cos. A}}{1-\text{cos. A}}} = \frac{1+\text{cos. A}}{\text{sin. A}} = \frac{\text{sin. A}}{1-\text{cos. A}} \quad (33)$$

$$\text{sec. } \tfrac{1}{2}\text{A} = \sqrt{\frac{2 \text{ sec. A}}{\text{sec. A}+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

$$\text{cosec. } \tfrac{1}{2}\text{A} = \sqrt{\frac{2 \text{ sec. A}}{\text{sec. A}-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

PROBLEM XVI.

To determine the sine of $(n+1)$A, *in terms of* nA, $(n-1)$A, *and* A.

Equations (1) and (2) may be thus written:

$$\text{sin. (B+A)} = \text{sin. B cos. A} + \text{sin. A cos. B};$$
$$\text{sin. (B}-\text{A)} = \text{sin. B cos. A} - \text{sin. A cos. B}.$$

Add these two equations, and we have

$$\text{sin. (B+A)} + \text{sin. (B}-\text{A)} = 2 \text{ sin. B cos. A}.$$

Subtract sin. (B$-$A) from each member, then

$$\text{sin. (B+A)} = 2 \text{ sin. B cos. A} - \text{sin. (B}-\text{A)}.$$

Let B$=n$A, then

$$\text{sin. } (n+1)\text{A} = 2 \text{ sin. } n\text{A cos. A} - \text{sin. } (n-1)\text{A} \quad . \quad (36)$$

In the above formula, let $n=1$; then $n+1=2$, $n-1=0$.

Therefore sin. 2A$=2$ sin. A cos. A$-$sin. 0;
$=2$ sin. A cos. A, the same as equat. (24).

Let $n=2$; then $n+1=3$, and $n-1=1$.

Therefore

$$\begin{aligned}\text{sin. 3A} &= 2 \text{ sin. 2A cos. A} - \text{sin. A};\\ &= 2\times 2 \text{ sin. A cos. A cos. A} - \text{sin. A};\\ &= 4 \text{ sin. A cos.}^2 \text{ A} - \text{sin. A};\\ &= 4 \text{ sin. A } (1-\text{sin.}^2 \text{ A}) - \text{sin. A};\\ &= 4 \text{ sin. A} - 4 \text{ sin.}^3 \text{ A} - \text{sin. A};\\ &= 3 \text{ sin. A} - 4 \text{ sin.}^3 \text{ A} \quad . \quad . \quad . \quad (37)\end{aligned}$$

Let $n=3$; then $n+1=4$, and $n-1=2$.

Therefore sin. 4A$=2$ sin. 3A cos. A$-$sin. 2A;

but, by the last equation, sin. 3A$=3$ sin. A-4 sin.^{3} A.

Hence $2\sin.3A=2(3\sin.A-4\sin.^3A)$, and

$$\begin{aligned}\sin.4A&=(6\sin.A-8\sin.^3A)\cos.A-2\sin.A\cos.A;\\&=\{(6-8\sin.^2A)\cos.A-2\cos.A\}\sin.A;\\&=\{(6-(8-8\cos.^2A)\cos.A-2\cos.A\}\sin.A;\\&=\{-2\cos.A+8\cos.^3A-2\cos.A\}\sin.A;\\&=\{8\cos.^3A-4\cos.A\}\sin.A\quad.\;.\;.\quad(38)\end{aligned}$$

In like manner we may continue the process for $\sin.5A$, $\sin.6A$, &c.

PROBLEM XVII.

To determine the cosine of $(n+1)A$ *in terms of* nA, $(n-1)A$, *and* A.

By equations (3), (4):

$$\cos.(B+A)=\cos.B\cos.A-\sin.B\sin.A;$$
$$\cos.(B-A)=\cos.B\cos.A+\sin.B\sin.A.$$

Adding these two equations, we have

$$\cos.(B+A)+\cos.(B-A)=2\cos.B\cos.A.$$

Subtract $\cos.(B-A)$ from both members, then

$$\cos.(B+A)=2\cos.B\cos.A-\cos.(B-A).$$

Let $B=nA$, we have

$$\cos.(n+1)A=2\cos.nA\cos.A-\cos.(n-1)A\quad.\quad(39)$$

Put $n=1$, then $n+1=2$, $n-1=0$.

Therefore $\cos.2A=2\cos.^2A-\cos.0;$

$=2\cos.^2A-1$, the same as (27).

Put $n=2$, then $n+1=3$, $n-1=1$, and

$$\begin{aligned}\cos.3A&=2\cos.2A\cos.A-\cos.A;\\&=2(2\cos.^2A-1)\cos.A-\cos.A;\\&=4\cos.^3A-2\cos.A-\cos.A;\\&=4\cos.^3A-3\cos.A\quad.\;.\;.\;.\quad(40)\end{aligned}$$

Put $n=3$, then $n+1=4$, $n-1=2$.

$$\begin{aligned}\cos.4A&=2\cos.3A\cos.A-\cos.2A;\\&=2(4\cos.^3A-3\cos.A)\cos.A-(2\cos.^2A-1);\\&=8\cos.^4A-6\cos.^2A-2\cos.^2A+1;\\&=8\cos.^4A-8\cos.^2A+1\quad.\;.\;.\;.\;.\quad(41)\end{aligned}$$

In like manner we may find $\cos.5A$, $\cos.6A$, &c.

(ART. 8.) By adding and subtracting (1) and (2), (3) and (4), we will have the following useful formulas:

$$\left.\begin{aligned}\sin.(A+B)+\sin.(A-B)&=2\sin.A\cos.B\\\sin.(A+B)-\sin.(A-B)&=2\sin.B\cos.A\\\cos.(A+B)+\cos.(A-B)&=2\cos.A\cos.B\\\cos.(A+B)-\cos.(A-B)&=-2\sin.A\sin.B\end{aligned}\right\}(42)$$

Multiply (1) and (2) together, we have

$$\sin.(A+B)\sin.(A-B)=\sin.^2A\cos.^2B-\sin.^2B\cos.^2A.$$

But

$\cos.^2B=1-\sin.^2B\ \therefore\ \sin.^2A\cos.^2B=\sin.^2A-\sin.^2A\sin.^2B;$

and

$\cos.^2A=1-\sin.^2A\ \therefore\ \sin.^2B\cos.^2A=\sin.^2B-\sin^2A\sin.^2B.$

Substitute these values in the above equations, then

$$\sin.(A+B)\sin.(A-B)=\sin.^2A-\sin.^2A\sin.^2B-(\sin.^2B-\sin.^2A\sin.^2B);$$

hence $\sin.(A+B)\sin.(A-B)=\sin.^2A-\sin.^2B$. (43)

Multiply (3) and (4), we shall have

$$\cos.(A+B)\cos.(A-B)=\cos.^2A\cos.^2B-\sin.^2A\sin.^2B.$$

But

$\cos.^2B=1-\sin.^2B\ \therefore\ \cos.^2A\cos.^2B=\cos.^2A-\cos.^2A\sin.^2B;$

and

$\sin.^2A=1-\cos.^2A\ \therefore\ \sin.^2A\sin.^2B=\sin.^2B-\cos.^2A\sin.^2B.$

Substitute these values in the above equation, and we have

$$\cos.(A+B)\cos.(A-B)=\cos.^2A-\cos.^2A\sin.^2B-(\sin.^2B-\cos.^2A\sin.^2B);$$

hence $\cos.(A+B)\cos.(A-B)=\cos.^2A-\sin.^2B;$

or $\cos.(A+B):\cos.A+\sin.B::\cos.A-\sin.B:\cos.(A-B)$ . . . . (44)

THEOREM III.

(ART. 9.) *In any plane triangle, the sum of any two sides is to their difference as the tangent of half the sum of the angles opposite [illegible] them is to the tangent of half their difference.*

(Theorem 2.) $\frac{a}{b}=\frac{\sin.A}{\sin.B}$; or $a:b=\sin.A:\sin.B$.

By the theory of proportions, we have

$$a+b:a-b=\sin.A+\sin.B:\sin.A-\sin.B.$$

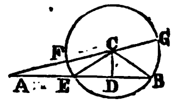
Whence $$\frac{a+b}{a-b}=\frac{\sin.A+\sin.B}{\sin.A-\sin.B}.$$

(Equation 17): $\frac{\sin.A+\sin.B}{\sin.A-\sin.B}=\frac{\tan.\frac{1}{2}(A+B)}{\tan.\frac{1}{2}(A-B)}$; or,

$$\left.\begin{aligned}&\frac{a+b}{a-b}=\frac{\tan.\frac{1}{2}(A+B)}{\tan.\frac{1}{2}(A-B)}.\\ \text{In like manner, }&\frac{a+c}{a-c}=\frac{\tan.\frac{1}{2}(A+C)}{\tan.\frac{1}{2}(A-C)}\\ &\frac{b+c}{b-c}=\frac{\tan.\frac{1}{2}(B+C)}{\tan.\frac{1}{2}(B-C)}\end{aligned}\right\}\ \ldots\ldots\ (45)$$

THEOREM IV.

In any plane triangle, the greatest side is to the sum of the other two sides as their difference is to the difference of the segments made by a perpendicular let fall from the angle opposite the greater side.



Demonstration. Let ABC be a plane triangle, of which the side AB is the greatest. With the centre C and radius$=$BC, describe the circle FEBG. Draw CD perpendicular to AB. Produce AC to meet the circle at G; draw CE.

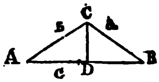
Then, because AB and AG are secants, it will be

As AB : AG :: AF : AE (Geom., B. II., Prop. 21). But AG$=$AC$+$CG$=$AC$+$BC, and AF$=$AC$-$CF$=$AC$-$BC, and AE$=$AD$-$DE$=$AD$-$DB. Substitute these values of AG, AE, and AF, in the above proposition, we have

$$AB : AC+BC :: AC-BC : AD-DB \quad . \quad (46)$$

THEOREM V.

In any plane triangle, the cosine of either of the angles is equal to the sum of the squares of its adjacent sides minus the square of the side opposite divided by twice the rectangle of the adjacent sides.



1°. Let ABC be a triangle, A, B, C the angles, and a, b, c the sides respectively. Let A, the proposed angle, be acute. From C draw CD perpendicular to AB; then

$\overline{BC}^2=\overline{AC}^2+\overline{AB}^2-2AB\times AD$ (Geom., B. I., Prop. 37);

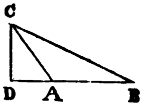
or $$a^2=b^2+c^2-2c\times AD.$$

But, because CDA is a right-angled triangle,

$$AD=AC \cos. A=b \cos. A;$$

$$\therefore a^2=b^2+c^2-2bc \cos. A;$$

$$\therefore \cos. A=\frac{b^2+c^2-a^2}{2bc}.$$



2°. Let A, the proposed angle, be obtuse: from C draw CD perpendicular to AB produced; then

$$\overline{BC}^2=\overline{AC}^2+\overline{AB}^2+2BA\times AD;$$

or $$a^2=b^2+c^2+2c\times AD.$$

But, since CDA is a right-angled triangle, AD=AC cos. CAD ∴ AD=AC × −cos. CAB (because CAB is the supplement of CAD).

$$\therefore AD=-b\cos.A;\ \text{hence } a^2=b^2+c^2-2bc\cos.A;$$

or $$\cos.A=\frac{b^2+c^2-a^2}{2bc}.$$

The two results are identical. Therefore, whether A be acute or obtuse, we shall have

$$\cos.A=\frac{b^2+c^2-a^2}{2bc} \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

To express the sine of an angle of a plane triangle in terms of the sides of the triangle.

Let A be the angle; then, by (47),

$$\cos.A=\frac{b^2+c^2-a^2}{2bc}.$$

Add unity to each member of the equation,

$$1+\cos.A=1+\frac{b^2+c^2-a^2}{2bc};$$

$$=\frac{b^2+2bc+c^2-a^2}{2bc};$$

$$=\frac{(b+c+a)(b+c-a)}{2bc} \quad . \quad . \quad (48)$$

Again: subtracting each member of (equation 47) from unity, we have

$$1-\cos.A=1-\frac{b^2+c^2-a^2}{2bc};$$

$$=\frac{2bc-b^2-c^2+a^2}{2bc};$$

$$=\frac{a^2-(b^2-2bc+c^2)}{2bc};$$

$$=\frac{(a+b-c)(a+c-b)}{2bc} \quad . \quad . \quad . \quad (49)$$

Multiply equations (48), (49), we shall have

$$1-\cos.^2A=\sin.^2A=\frac{(a+b+c)(b+c-a)(a+b-c)(a+c-b)}{4b^2c^2}.$$

$$\therefore \sin. A=\frac{1}{2bc}\sqrt{(a+b+c)(b+c-a)(a+b-c)(a+c-b)} \quad (50)$$

The above equation may be expressed somewhat differently, thus:

Let $s=\frac{1}{2}(a+b+c)$; then $2s=a+b+c$. Now subtract $2a$ from both members, we have

$$2s-2a=b+c-a;$$

or

$$s-a=\frac{b+c-a}{2}.$$

In like manner, $s-b=\frac{a+c-b}{2}$;

$$s-c=\frac{a+b-c}{2}.$$

Substitute $2s$, $2(s-a\,..)$, &c., in the expression for $\sin.^2 A$, it becomes

$$\sin.^2 A=\frac{16s(s-a)(s-b)(s-c)}{4b^2c^2};$$

therefore $\sin. A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$ . . . (51)

To express the sine, cosine, and tangent of half an angle of a plane triangle in terms of the sides of the triangle.

(Equation 49): $1+\cos. A=\frac{(a+b+c)(b+c-a)}{2bc}$;

$$=\frac{4s(s-a)}{2bc};$$

but $1+\cos. A=2\cos.^2 \frac{1}{2}A$;

hence $2\cos.^2 \frac{1}{2}A=\frac{4s(s-a)}{2bc}$;

$$\therefore \cos. \tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}} \quad . \ . \ . \ . \ (52)$$

In like manner,

$$\cos. \tfrac{1}{2}B=\sqrt{\left(\frac{s(s-b)}{ac}\right)} \quad . \ . \ . \ . \ (53)$$

$$\cos. \tfrac{1}{2}C=\sqrt{\left(\frac{s(s-c)}{ab}\right)} \quad . \ . \ . \ . \ (54)$$

Again (25): $1-\cos. A=2\sin.^2 \frac{1}{2}A$;

$$\therefore 2\ \sin.^2 \tfrac{1}{2}A=\frac{4(s-b)(s-c)}{2bc};$$

hence
$$\sin. \tfrac{1}{2}A=\sqrt{\left(\frac{(s-b)(s-c)}{bc}\right)} \quad . \quad . \quad . \quad (55)$$

In like manner,

$$\sin. \tfrac{1}{2}B=\sqrt{\left(\frac{(s-a)(s-c)}{ac}\right)} \quad . \quad . \quad . \quad (56)$$

$$\sin. \tfrac{1}{2}C=\sqrt{\left(\frac{(s-a)(s-b)}{ab}\right)} \quad . \quad . \quad . \quad (57)$$

By dividing (55), (56), (57), by (52), (53), (54), we obtain

$$\tan. \tfrac{1}{2}A=\sqrt{\left(\frac{(s-b)(s-c)}{s(s-a)}\right)} \quad . \quad . \quad . \quad . \quad (58)$$

$$\tan. \tfrac{1}{2}B=\sqrt{\left(\frac{(s-a)(s-c)}{s(s-b)}\right)} \quad . \quad . \quad . \quad . \quad (59)$$

$$\tan. \tfrac{1}{2}C=\sqrt{\left(\frac{(s-a)(s-b)}{s(s-c)}\right)} \quad . \quad . \quad . \quad . \quad (60)$$

CHAPTER III.

(ART. 10.) *Construction of Trigonometrical Tables.*

It is manifest, from Definitions (1), (2), (3), &c., that the various trigonometrical quantities, the *sine*, *cosine*, *tangent*, &c., are abstract numbers, representing the comparative lengths of certain lines. In Chapter II. the numerical values of some of these quantities have been found: it is the purpose of this to show how the numbers corresponding to angles of every degree of magnitude may be obtained by the application of simple principles.

The first operation to be performed is

1°. *To compute the numerical value of the sine of* 1′.

The *arc* of one minute is $\frac{1}{10800}$ of the semi-circumference, whose radius is unity. The length of this semi-circumference has been found to be 3.1415926+. Therefore the

$$\sin. 1'=\frac{3.1415926}{108000}=0.000290888 \quad . \quad . \quad (61)$$

$$\text{sin. } 2' = \frac{3.1415926}{5400} = 0.000581776 \quad . \quad . \quad (62)$$

$$\text{\&c.} = \qquad \text{\&c.} = \qquad \text{\&c.}$$

$$\text{sin. } 1^\circ = \frac{3.1415926}{180} = 0.01745 \quad . \quad . \quad . \quad (63)$$

2°. *Having the sine of* 1°, *to find the sine of* 2°, 3°, 4°, 5°, 6°, 7°, 8°, 9°, 10°.

Equation (18):

$$\text{sin. } 2A = 2 \text{ sin. } A \text{ cos. } A = 2 \text{ sin. } A\sqrt{1 - \text{sin.}^2 A}.$$

Put $A = 1^\circ$; then

$$\text{sin. } 2^\circ = 2 \text{ sin. } 1^\circ\sqrt{1 - \text{sin.}^2 1^\circ} = 0.034895 \quad . \quad (64)$$

Equation (40):

$$\text{sin. } (A+B) \text{ sin. } (A-B) = \text{sin.}^2 A - \text{sin.}^2 B;$$

$$= (\text{sin. } A + \text{sin. } B)(\text{sin. } A - \text{sin. } B);$$

or, $\text{sin. } (A-B) : \text{sin. } A - \text{sin. } B :: \text{sin. } A + \text{sin. } B : \text{sin. } (A+B)$.

That is, *the sine of the difference of any two angles is to the difference of their sines as the sum of those sines is to the sine of their sum.*

Therefore, put $A = 2^\circ$, $B = 1^\circ$; then

$\text{sin. } 1^\circ : \text{sin. } 2^\circ - \text{sin. } 1^\circ :: \text{sin. } 2^\circ + \text{sin. } 1^\circ : \text{sin. } 3^\circ = .052336$.

Put $A = 3^\circ$, $B = 1^\circ$:

$\text{sin. } 2^\circ : \text{sin. } 3^\circ - \text{sin. } 1^\circ :: \text{sin. } 3^\circ + \text{sin. } 1^\circ : \text{sin. } 4^\circ = .069756$.

By assuming $A = 4^\circ$, 5°, &c., $B = 1^\circ$, we obtain the sines required. Using the same proportions, knowing the sines of 5°, 10°, and 15°, the sines of 20°, 25°, 35°, 55°, 65°, &c., may be found.

Equation (25):

$$\text{sin. } \tfrac{1}{2}A = \sqrt{\left(\frac{1 - \text{cos. } A}{2}\right)} = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 A}}.$$

Substitute $\frac{1}{2}A$ for A, and it becomes

$$\text{sin. } \tfrac{1}{4}A, \text{ or sin. } \tfrac{1}{2_2}A = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 \tfrac{1}{2}A}};$$

$$\text{sin. } \tfrac{1}{2_3}A = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 \tfrac{1^2}{2}A}};$$

and generally,
$$\text{sin. } \tfrac{1}{2_n}A = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 \tfrac{1}{2_{n-1}}A}}.$$

Assume $A = 30^\circ$, then $\frac{1}{2}A = 15^\circ$; and applying the above formula, we have

$$\text{sin. } 15^\circ = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \text{sin.}^2 30^\circ}}.$$

But $\sin. 30^\circ=\frac{1}{2}$, and $\sin.^2 30^\circ=\frac{1}{4}$;

$$\therefore \sin. 15^\circ=\sqrt{\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-\tfrac{1}{4}}}=0.258819.$$

In like manner,

$$\begin{aligned}\sin. 7^\circ\ 30'&=\sqrt{\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-\sin.^2 15^\circ}}\\ &=\sqrt{\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-(.258819)^2}}\\ &=0.1305268,\\ &\quad \&c., \quad \&c.\end{aligned}$$

3°. Again, from equation (1): $\sin. (A+B)=\sin. A \cos. B+\sin. B \cos. A$.

Let $B=1'$; then

$$\sin. (A+1')=\sin. A \cos. 1'+\sin. 1' \cos. A;$$

but $\cos. 1'=1$ and $\sin. 1'=0.00029$. Therefore,

$$\left.\begin{aligned}\sin. (A+1')&\doteq\sin. A+\sin. 1' \cos. A\\ &=\sin. A+.00029 \cos. A\end{aligned}\right\} \quad . \quad (65)$$

By equation (65) the sine of an angle of any number of degrees and minutes may be calculated.

(Art. 11.) *To find the sine, &c., of* 0° *and* 90°.

Formula (1): $\sin. (A-B)=\sin. A \cos. B-\sin. B \cos. A$.

Put $B=A$; we have

$$\sin. 0^\circ=\sin. A \cos. A-\sin. A \cos. A=0;$$
$$\sin. 0^\circ=\cos. 90^\circ=0.$$

Formula (4): $\cos. (A-B)=\cos. A \cos. B+\sin. A \sin. B$.

Put $B=A$; then

$$\begin{aligned}\cos. 0^\circ&=\cos. A \cos. A+\sin. A \sin. A;\\ &=\cos.^2 A+\sin.^2 A=1;\end{aligned}$$
$$\cos. 0^\circ=\sin. 90^\circ=1.$$

$$\tan. 0^\circ=\cot. 90^\circ=\frac{\sin. 0^\circ}{\cos. 0^\circ}=\frac{0}{1}=0;$$

$$\cot. 0^\circ=\tan. 90^\circ=\frac{\cos. 0^\circ}{\sin. 0^\circ}=\frac{1}{0}=\infty;$$

$$\sec. 0^\circ=\operatorname{cosec.} 90^\circ=\frac{1}{\cos. 0^\circ}=\frac{1}{1}=1;$$

$$\operatorname{cosec.} 0^\circ=\sec. 90^\circ=\frac{1}{\sin. 0^\circ}=\frac{1}{0}=\infty.$$

To find the sine of 180°.

Formula (24): $\sin. 180° = 2 \sin. 90° \cos. 90° = 0$ (72)

(26) $\cos. 180° = \cos.^2 90° - \sin.^2 90 = -1$. (73)

$$\tan. 180° = \frac{\sin. 180°}{\cos. 180°} = \frac{0}{-1} = 0 \quad . \; . \; . \quad (74)$$

$$\cot. 180° = \frac{\cos. 180°}{\sin. 180°} = \frac{-1}{0} = -\infty \quad . \; . \quad (75)$$

$$\sec. 180° = \frac{1}{\cos. 180°} = \frac{1}{-1} = -1 \quad . \; . \quad (76)$$

$$\operatorname{cosec.} 180° = \frac{1}{\sin. 180°} = \frac{1}{0} = \infty \quad . \; . \; . \quad (77)$$

(Art. 12.) *To find the sine, &c., of* 270°.

In formula (1), make $A = 180°$, $B = 90°$; then

$\sin. (180° + 90°) = \sin. 180° \cos. 90° + \sin. 90° \cos. 180°$;

$\sin. 270° = -1$ . . . . (78)

$\cos. (180° + 90°) = \cos. 180° \cos. 90° - \sin. 180° \sin. 90°$;

$\cos. 270° = 0$ . . . . (79)

$$\tan. 270° = \frac{\sin. 270°}{\cos. 270°} = \frac{-1}{0} = -\infty \quad . \; . \quad (80)$$

$$\cot. 270° = \frac{\cos. 270°}{\sin. 270°} = \frac{0}{-1} = 0 \quad . \; . \quad (80)$$

$$\sec. 270° = \frac{1}{\cos. 270°} = \frac{1}{0} = \infty \quad . \; . \quad (81)$$

$$\operatorname{cosec.} 270° = \frac{1}{\sin. 270°} = \frac{1}{-1} = -1 \quad . \; . \quad (82)$$

To find the sine, &c., of 360°.

In formula (18), make $A = 180°$; then

$\sin. 2A = 2 \sin. A \cos. A$ becomes

$\sin. 360° = 2 \sin. 180° \cos. 180° = 0$ (83)

(26) $\cos. 360° = \cos.^2 180° - \sin.^2 180° = 1 = \cos. 0°$ (84)

Therefore the sine, &c., of 360° are the same as those of 0°.

To find the sine, &c., of an angle greater than 360°.

By formula (1):

$\sin. (360° + A) = \sin. 360° \cos. A + \sin. A \cos. 360°$

$= \sin. A$ . . . . (85)

(ART. 13.) Hence the sine, &c., of an angle greater than 360° are equal to the sine, &c., of its excess above 360°.

To find the sine, &c., of a negative angle.

In formula (2), make $A=0$; then

$$\sin.(-B)=\sin.0\ \cos.B-\sin.B\ \cos.0$$
$$=-\sin.B.$$
&c., &c.

QUESTIONS FOR EXERCISE.

1. If θ be any angle, prove that $\tan.^2\theta-\sin.^2\theta=\tan.^2\theta$ $\sin.^2\theta$. But $1-\sin.^2\theta=\cos.^2\theta$ (from Eq. 1, Chap. I.).

Multiply each member of this equation by $\sin.^2\theta$, and it becomes

$$\sin.^2\theta-\sin.^2\theta\ \sin.^2\theta=\sin.^2\theta\ \cos.^2\theta;$$

or, by transposition,

$$\sin.^2\theta-\sin.^2\theta\ \cos.^2\theta=\sin.^2\theta\ \sin.^2\theta.$$

Divide by $\cos.^2\theta$, we have

$$\frac{\sin.^2\theta}{\cos.^2\theta}-\sin.^2\theta=\frac{\sin.^2\theta}{\cos.^2\theta}\sin.^2\theta.$$

But
$$\frac{\sin.^2\theta}{\cos.^2\theta}=\tan.^2\theta.$$
$$\therefore \tan.^2\theta-\sin.^2\theta=\tan.^2\theta\ \sin.^2\theta.$$

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2. Prove the trigonometrical formulas:

$$\sin.\tfrac{3}{2}\theta=\sin.\tfrac{1}{2}\theta(2\ \cos.\theta+1);$$
$$\cos.\tfrac{3}{2}\theta=\cos.\tfrac{1}{2}\theta(2\ \cos.\theta-1);$$
$$\tan.\tfrac{3}{2}\theta=\frac{\sin.\theta+\sin.2\theta}{\cos.\theta+\cos.2\theta}.$$

1°. $\sin.\tfrac{3}{2}\theta=\sin.(\theta+\tfrac{1}{2}\theta)=\sin.\theta\ \cos.\tfrac{1}{2}\theta+\sin.\tfrac{1}{2}\theta\ \cos.\theta;$

$$=2\ \sin.\tfrac{1}{2}\theta\ \cos.^2\tfrac{1}{2}\theta+\sin.\tfrac{1}{2}\theta\ \cos.\theta;$$
$$=\sin.\tfrac{1}{2}\theta(2\ \cos.^2\tfrac{1}{2}\theta+\cos.\theta).$$

But $2\ \cos.^2\tfrac{1}{2}\theta=1+\cos.\theta;$

$$\therefore \sin.\tfrac{3}{2}\theta=\sin.\tfrac{1}{2}\theta(2\ \cos.\theta+1).$$

2°. $\cos.\tfrac{3}{2}\theta=\cos.(\theta+\tfrac{1}{2}\theta)=\cos.\theta\ \cos.\tfrac{1}{2}\theta-\sin.\theta\ \sin.\tfrac{1}{2}\theta;$

$$=\cos.\theta\ \cos.\tfrac{1}{2}\theta-2\ \cos.\tfrac{1}{2}\theta\ \sin.^2\tfrac{1}{2}\theta$$
$$=\cos.\tfrac{1}{2}\theta(\cos.\theta-2\ \sin.^2\tfrac{1}{2}\theta).$$

But $2\ \sin.^2\tfrac{1}{2}\theta=1-\cos.\theta.$

$$\therefore \cos.\tfrac{3}{2}\theta=\cos.\tfrac{1}{2}\theta(2\ \cos.\theta-1).$$

3°. $$\tan.\tfrac{3}{2}\theta=\frac{\sin.\tfrac{3}{2}\theta}{\cos.\tfrac{3}{2}\theta}=\frac{\sin.\tfrac{1}{2}\theta(2\cos.\theta+1)}{\cos.\tfrac{1}{2}\theta(2\cos.\theta-1)}$$

$$=\frac{2\sin.\tfrac{1}{2}\theta\cos.\tfrac{1}{2}\theta(2\cos.\theta+1)}{2\cos.^2\tfrac{1}{2}\theta(2\cos.\theta-1)}$$

$$=\frac{\sin.\theta(2\cos.\theta+1)}{(1+\cos.\theta)(2\cos.\theta-1)}$$

$$=\frac{\sin.\theta+2\sin.\theta\cos.\theta}{\cos.\theta+2\cos.^2\theta-1}=\frac{\sin.\theta+\sin.2\theta}{\cos.\theta+\cos.2\theta}.$$

3. Eliminate θ from the two following equations, and find the value of x.

$$x\cos.\theta+a\sin.\theta=b \quad . \quad . \quad . \quad . \quad (1)$$

$$x\sin.\theta-a\cos.\theta=c \quad . \quad . \quad . \quad . \quad (2)$$

Square the two equations, and we have

$$x^2\cos.^2\theta+2ax\sin.\theta\cos.\theta+a^2\sin.^2\theta=b^2 \quad . \quad (3)$$

$$x^2\sin.^2\theta-2ax\sin.\theta\cos.\theta+a^2\cos.^2\theta=c^2 \quad . \quad (4)$$

Add (3) and (4) together, we have

$$x^2(\sin.^2\theta+\cos.^2\theta)+a^2(\sin.^2\theta+\cos.^2\theta)=b^2+c^2.$$

But $\sin.^2\theta+\cos.^2\theta=1.$

$$\therefore x^2+a^2=b^2+c^2;$$

whence $x=\sqrt{b^2+c^2-a^2}.$

4. In a plane triangle, given

$$b=a\sin.\mathrm{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$c=a\cos.\mathrm{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

to find the angles.

Multiply equation (2) by $2c$, we have

$$2c^2=2ac\cos.\mathrm{B} \quad . \quad . \quad . \quad . \quad (3)$$

By (47): $b^2=a^2+c^2-2ac\cos.\mathrm{B}$; add these, we have

$$b^2+c^2=a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence the triangle is right-angled, a being the hypothenuse; then $\mathrm{A}=90°$, and $\mathrm{B}+\mathrm{C}=90°$;

$$\sin.\mathrm{C}=\cos.\mathrm{B};$$

$$b=a\sin.\mathrm{C},\ c=a\cos.\mathrm{B};$$

$$b=a.$$

Hence the triangle is isosceles $\therefore \mathrm{A}=90°$, $\mathrm{B}=45°=\mathrm{C}$ $=45°$.

5. If θ be any angle, prove that

$$\tan.^2\theta-\tan.^2\tfrac{1}{3}\theta=\frac{8\sin.^2\tfrac{1}{3}\theta\cos.\tfrac{2}{3}\theta}{\cos.^2\theta}.$$

Solution:

$$\tan.^2\theta-\tan.^2\tfrac{1}{3}\theta=(\tan.\theta+\tan.\tfrac{1}{3}\theta)(\tan.\theta-\tan.\tfrac{1}{3}\theta);$$

but,
$$\tan.\,\theta+\tan.\,\tfrac{1}{3}\theta=\frac{\sin.\,\theta}{\cos.\,\theta}+\frac{\sin.\,\frac{1}{3}\theta}{\cos.\,\frac{1}{3}\theta};$$
$$=\frac{\sin.\,\theta\cos.\,\frac{1}{3}\theta+\cos.\,\theta\sin.\,\frac{1}{3}\theta}{\cos.\,\theta\cos.\,\frac{1}{3}\theta};$$
$$=\frac{\sin.\,(\theta+\frac{1}{3}\theta)}{\cos.\,\theta\cos.\,\frac{1}{3}\theta};$$
$$=\frac{\sin.\,\frac{4}{3}\theta}{\cos.\,\theta\cos.\,\frac{1}{3}\theta};$$
$$=\frac{4\sin.\,\frac{1}{3}\theta\cos.\,\frac{1}{3}\theta\cos.\,\frac{2}{3}\theta}{\cos.\,\theta\cos.\,\frac{1}{3}\theta};$$
$$=\frac{4\sin.\,\frac{1}{3}\theta\cos.\,\frac{2}{3}\theta}{\cos.\,\theta};$$

and
$$\tan.\,\theta-\tan.\,\tfrac{1}{3}\theta=\frac{\sin.\,\theta}{\cos.\,\theta}-\frac{\sin.\,\frac{1}{3}\theta}{\cos.\,\frac{1}{3}\theta};$$
$$=\frac{\sin.\,\frac{2}{3}\theta}{\cos.\,\theta\cos.\,\frac{1}{3}\theta};$$
$$=\frac{2\sin.\,\frac{1}{3}\theta}{\cos.\,\theta}.$$

Therefore
$$\tan.^2\theta-\tan.^2\tfrac{1}{3}\theta=\frac{8\sin.^2\frac{1}{3}\theta\cos.\,\frac{2}{3}\theta}{\cos.^2\theta}.$$

6. Prove that the sum of the tangents of the three angles in every plane triangle is equal to their product.

Solution. If θ, θ', θ'' be the three angles, then $\theta+\theta'+\theta''=180°$;

or
$$\tan.\,(180°-\theta)=\tan.\,(\theta'+\theta'').$$

Therefore
$$-\tan.\,\theta=\frac{\tan.\,\theta'+\tan.\,\theta''}{1-\tan.\,\theta'\tan.\,\theta''};$$

hence $-\tan.\,\theta+\tan.\,\theta\tan.\,\theta'\tan.\,\theta''=\tan.\,\theta'+\tan.\,\theta''$;

or $\tan.\,\theta+\tan.\,\theta'+\tan.\,\theta''=\tan.\,\theta\tan.\,\theta'\tan.\,\theta''$.

7. Given $a\sin.\,\theta+b\cos.\,\theta=c$, to find θ.

Ans. $\sin.\,\theta=\dfrac{ac\pm b\sqrt{a^2+b^2-c^2}}{a^2+b^2}$.

8. If the tan. of an arc be equal to $\sqrt{n}$, prove that the sine of the same arc is $=\sqrt{\dfrac{n}{n+1}}$.

9. Eliminate θ from the two equations,

$$x \cos.\theta + a \sin.\theta = b,$$
$$x \sin.\theta + a \cos.\theta = c.$$

Ans. $\dfrac{(b+c)^2}{(x+a)^2} + \dfrac{(b-c)^2}{(x-a)^2} = 2.$

10. Given $\tan.\theta + \cot.\theta = k$, to find θ.

Ans. $\tan.\theta = \frac{1}{2} \pm \frac{1}{2}\sqrt{k^2-4}.$

11. Given $k \cos.\theta = \tan.\theta$, to find θ.

Ans. $\cos.\theta = \sqrt{-\frac{1}{2k^2} \pm \frac{1}{k^2}\sqrt{4k^2+1}}.$

12. Prove the formula

$$\tan.\tfrac{1}{2}\theta + \cot.\tfrac{1}{2}\theta = 2 \operatorname{cosec}.\theta.$$

13. Find the angle ϕ, from the equation

$\dfrac{1+a\cos.(\phi+\theta)}{\sin.(\phi+\theta')} = \dfrac{1+a\cos.\theta}{\sin.\theta'}$, θ and θ' and a being known.

Ans. $\tan.\frac{1}{2}\phi = \cot.\theta' + a\dfrac{\cos.(\theta-\theta')}{\sin.\theta'}.$

14. In any plane triangle ABC, if the angle made by a line drawn from the angle C to the middle point of the side c be denoted by θ, prove that

$$2 \cot.\theta = \cot.A - \cot.B.$$

15. If the angles A, B, C of a plane triangle be as the numbers 2, 3, 4 respectively, prove that $2\cos.\frac{1}{2}A = \dfrac{a+c}{b}.$

(Art. 14.) *The following formulas, which are sometimes called the formulas of verification, are very useful.*

Since $\cos.^2 a + \sin.^2 a = 1,$

and $2 \sin.a \cos.a = \sin.2a,$

by adding and subtracting, we have

$$\cos.^2 a + 2\sin.a \cos.a + \sin.^2 a = 1 + \sin.2a,$$

and $\cos.^2 a - 2\sin.a \cos.a + \sin.^2 a = 1 - \sin.2a.$

By extracting the square root of each of these, we have

$$\cos.a + \sin.a = \sqrt{(1+\sin.2a)};$$
$$\cos.a - \sin.a = \sqrt{(1-\sin.2a)}.$$

When a is less than 45°, sin. a is less than cos. a; therefore, by adding,

$$2\cos.a = \sqrt{(1+\sin.2a)} + \sqrt{(1-\sin.2a)};$$
$$\therefore \cos.a = \tfrac{1}{2}\sqrt{(1+\sin.2a)} + \tfrac{1}{2}\sqrt{(1-\sin.2a)}.$$

Subtracting, $2 \sin. a = \sqrt{(1+\sin. 2a)} - \sqrt{(1-\sin. 2a)}$;

$$\therefore \sin. a = \tfrac{1}{2}\sqrt{(1+\sin. 2a)} - \tfrac{1}{2}\sqrt{(1-\sin. 2a)}.$$

When a is greater than 45°, then $\cos. a$ is less than $\sin. a$, and the above will become

$$\cos. a = \tfrac{1}{2}\{\sqrt{(1+\sin. 2a)} - \sqrt{(1-\sin. 2a)}\} \quad . \quad . \quad (A)$$
$$\sin. a = \tfrac{1}{2}\{\sqrt{(1+\sin. 2a)} + \sqrt{(1-\sin. 2a)}\} \quad . \quad . \quad (B)$$

1. *To find the sine of* 18°.

Since the cosine of an angle is equal to the sine of its complement, $\cos. 54° = \sin. 36°$; but

$$\cos. 54° = \cos. (36°+18°) = \cos. 36° \cos. 18° - \sin. 36° \sin. 18°;$$
$$\therefore \cos. 36° \cos. 18° - \sin. 36° \sin. 18° = \sin. 36°;$$

but $\cos. 36° = 1 - 2 \sin.^2 18°$,

and $\sin. 36° = 2 \sin. 18° \cos. 18°$.

$$\therefore \cos. 18°(1-2\sin.^2 18°) - 2\sin.^2 18° \cos. 18° = 2 \sin. 18° \cos. 18°;$$
$$\therefore 1 - 2\sin.^2 18° - 2\sin.^2 18° = 2\sin. 18°;$$
$$\therefore 4\sin.^2 18° + 2\sin. 18° = 1.$$
$$\therefore \sin. 18° = \frac{\sqrt{5}-1}{4}.$$

2. Prove that $\cos. (60°+a) + \cos. (60°-a) = \cos. a$.

1°. $\cos. (60°+a) = \cos. 60° \cos. a - \sin. 60° \sin. a$.

2°. $\cos. (60°-a) = \cos. 60° \cos. a + \sin. 60° \sin. a$.

$$\therefore \cos. (60°+a) + \cos. (60°-a) = 2 \cos. 60° \cos. a$$
$$= 2 \times \tfrac{1}{2} \cos. a = \cos. a;$$

because $\cos. 60° = \frac{1}{2}$. (Trig., Art. 4.)

3. Prove that $\sin. (30°+a) + \sin. (30°-a) = \cos. a$.

4. Prove that $\sin. a = \dfrac{2 \tan. \frac{1}{2}a}{1+\tan.^2 \frac{1}{2}a}$.

5. Prove that $2 \cos. (45°+\frac{1}{2}a) \cos. (45°-\frac{1}{2}a) = \cos. a$.

6. Given $\sin. (x+a) + \cos. (x+a) = \sin. (x-a) + \cos. (x-a)$, to find $\sin. x$. *Ans.* $\sin. x = \frac{1}{2}\sqrt{2}$.

7. Given $\sin, x + \cos. 2x = \frac{1}{2}\sqrt{5}$, to find $\sin. x$.

Ans. $\sin. x = \dfrac{\sqrt{5}-1}{4}$.

8. Given $4 \sin. x \sin. 3x = 1$, to find $\sin. x$.

Ans. $\sin. x = \dfrac{\sqrt{5}\pm 1}{4}$.

9. Prove that $\dfrac{2 \sin. a - \sin. 2a}{2 \sin. a + \sin. 2a} = \tan.^2 \frac{1}{2}a$.

10. If $a+b+c=90°$, prove that

$$\cot. a+\cot. b+\cot. c=\cot. a \cot. b \cot. c.$$

11. If $a+b+c=180°$, prove that

$$\tan. a+\tan. b+\tan. c=\tan. a \tan. b \tan. c.$$

12. Find the tangent of 15°.

$$\tan. 15°=\tan. (45°-30°)=\frac{\tan. 45°-\tan. 30°}{1+\tan. 45° \tan. 30°}$$

$$=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}.$$

CHAPTER IV.

INVERSE TRIGONOMETRICAL FUNCTIONS.

(Art. 15.) The quantities, $\sin.^{-1} a$, $\cos.^{-1} a$, &c., are called *inverse functions*, and have the following signification:

(1.) $x=\sin.^{-1} a$ is read, x is equal to an arc or angle whose sine is a.

(2.) $x=\cos.^{-1} a$ is read, x is equal to an arc whose cosine is a.

(3.) $x=\tan.^{-1} a$ is read, x is equal to an arc whose tangent is a.

From (1): $\sin. x=a$; (2) $\cos. x=a$; (3) $\tan. x=a$.

Let $\sin. a=p$, and $\sin. b=q$;

then $a=\sin.^{-1} p$, and $b=\sin.^{-1} q$;

and, since $\sin. a=p$, $\therefore \cos. a=\sqrt{1-p^2}$, $\cos. b=\sqrt{1-q^2}$,

$$\sin. (a+b)=\sin. a \cos. b+\cos. a \sin. b;$$

$$=p\sqrt{(1-q^2)}+q\sqrt{(1-p^2)}.$$

$$\therefore (a+b)=\sin.^{-1} \{p\sqrt{(1-q^2)}+q\sqrt{(1-p^2)}\};$$

but $a=\sin.^{-1} p$, and $b=\sin.^{-1} q$.

$$\therefore \sin.^{-1} p+\sin.^{-1} q=\sin.^{-1} \{p\sqrt{(1-q^2)}+q\sqrt{(1-p^2)}\}.$$

In like manner,

$$\sin.^{-1} p-\sin.^{-1} q=\sin.^{-1} \{p\sqrt{(1-q^2)}-q\sqrt{(1-p^2)}\}.$$

By proceeding in a similar manner for the cosine,

$$\cos.^{-1} p+\cos.^{-1} q=\cos.^{-1} \{pq-\sqrt{(1-p^2)}\sqrt{(1-q^2)}\};$$

$$\cos.^{-1} p-\cos.^{-1} q=\cos.^{-1} \{pq+\sqrt{(1-p^2)}\sqrt{(1-q^2)}\}.$$

Proceeding in the same manner for the tangents:

If $\qquad \text{tan.}\, a=p, \text{ and } \tan b=q,$

then $\qquad a=\text{tan.}^{-1} p, \text{ and } b=\text{tan.}^{-1} q;$

$$\text{tan.}\,(a+b)=\frac{\text{tan.}\, a+\text{tan.}\, b}{1-\text{tan.}\, a\ \text{tan.}\, b}=\frac{p+q}{1-pq}.$$

$$\therefore a+b=\text{tan.}^{-1}\left(\frac{p+q}{1-pq}\right).$$

But $\qquad a=\text{tan.}^{-1} p, \text{ and } b=\text{tan.}^{-1} q;$

$$\therefore \text{tan.}^{-1} p+\text{tan.}^{-1} q=\text{tan.}^{-1}\left(\frac{p+q}{1-pq}\right).$$

And since $\quad \text{tan.}\,(a-b)=\dfrac{\text{tan.}\, a-\text{tan.}\, b}{1+\text{tan.}\, a\ \text{tan.}\, b}=\dfrac{p-q}{1+pq};$

$$\therefore \text{tan.}^{-1} p-\text{tan.}^{-1} q=\text{tan.}^{-1}\left(\frac{p-q}{1+pq}\right).$$

In like manner,

$$\text{cot.}^{-1} p\pm\text{cot.}^{-} q=\text{cot.}^{-1}\left(\frac{pq\mp 1}{q\pm p}\right).$$

1. If $x=\text{sin.}^{-1}(\frac{3}{5})$, and $y=\text{sin.}^{-1}(\frac{4}{5})$, then $x+y=90°$.

For $\qquad \text{sin.}\, x=\frac{3}{5};\ \text{cos.}\, x=\sqrt{(1-\frac{9}{25})}=\frac{4}{5};$

and $\qquad \text{sin.}\, y=\frac{4}{5};\ \text{cos.}\, y=\sqrt{(1-\frac{16}{25})}=\frac{3}{5};$

$$\begin{aligned}\text{sin.}\,(x+y)&=\text{sin.}\, x\ \text{cos.}\, y+\text{cos.}\, x\ \text{sin.}\, y;\\ &=\tfrac{3}{5}\times\tfrac{3}{5}+\tfrac{4}{5}\times\tfrac{4}{5};\\ &=\tfrac{9}{25}+\tfrac{16}{25}=1.\end{aligned}$$

2. Prove that

$$\text{tan.}^{-1}\tfrac{1}{3}+\text{tan.}^{-1}\tfrac{1}{5}+\text{tan.}^{-1}\tfrac{1}{7}+\text{tan.}^{-1}\tfrac{1}{8}=45°.$$

$$(\text{tan.}^{-1}\tfrac{1}{3}+\text{tan.}^{-1}\tfrac{1}{5})+(\text{tan.}^{-1}\tfrac{1}{7}+\text{tan.}^{-1}\tfrac{1}{8})=\text{1st mem.};$$

but $\quad \text{tan.}^{-1}\frac{1}{3}+\text{tan.}^{-1}\frac{1}{5}=\text{tan.}^{-1}\left\{\dfrac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\cdot\frac{1}{5}}\right\}=\text{tan.}^{-1}\frac{4}{7},$

and $\quad \text{tan.}^{-1}\frac{1}{7}+\text{tan.}^{-1}\frac{1}{8}=\text{tan.}^{-1}\left\{\dfrac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\cdot\frac{1}{8}}\right\}=\text{tan.}^{-1}\frac{3}{11};$

$$\therefore \text{tan.}^{-}\tfrac{1}{3}+\text{tan.}^{-1}\tfrac{1}{5}+\text{tan.}^{-1}\tfrac{1}{7}+\text{tan.}^{-1}\tfrac{1}{8}=\text{tan.}^{-1}\tfrac{4}{7}+\text{tan.}^{-1}\tfrac{3}{11}.$$

But

$$\text{tan.}^{-1}\tfrac{4}{7}+\text{tan.}^{-1}\tfrac{3}{11}=\text{tan.}^{-1}\left\{\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\cdot\frac{3}{11}}\right\}=\text{tan.}^{-1}1=45°.$$

3. Prove that $\text{tan.}^{-1}\frac{1}{7}+2\ \text{tan.}^{-1}\frac{1}{3}=45°$.

4. Given $\text{tan.}^{-1}\left(\dfrac{1}{x-1}\right)-\text{tan.}^{-1}\left(\dfrac{1}{x+1}\right)=\dfrac{\pi}{12}$, to find x.

Ans. $x=\pm\dfrac{2}{\sqrt{3}-1}$.

5. Prove that

$$\tan.^{-1}\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)-\tan.^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}.$$

6. Given $\sec.^{-1}a-\sec.^{-1}b=\sec.^{-1}\frac{x}{b}-\sec.^{-1}\frac{x}{a}$, to find x.

Ans. $x=\pm ab$.

SOLUTION OF

RIGHT-ANGLED PLANE TRIANGLES

BY MEANS OF LOGARITHMS.

THE solution of right-angled triangles may be embraced in four cases:

1°. *When the hypothenuse and one of the acute angles are given.*

Chap. II., Theor. I. By multiplying the first member by radius to make the equation homogeneous,

$$R.BC=AC.\sin. A \ (1),$$

∴ (*a.*) *As radius is to the sine of the acute angle, so is the hypothenuse to the side opposite;* and,

(*b.*) *As radius* : *the cosine of the acute angle* :: *the hypothenuse to the side adjacent.*

2°. *When the hypothenuse and one side are given.*

From the above equation (1) we have,

(*c.*) *As the hypothenuse* : *the given side* :: *radius* : *the sine of the angle opposite the given side.*

(*d.*) *As radius* : *the sine of the angle opposite the required side* :: *the hypothenuse* : *the side required.*

3°. *When a side and one of the acute angles are given.*

(Chap. II., Theor. I.)

$$BC=AB.\tan. A,$$

or

$$R.BC=AB.\tan. A.$$

(*e.*) *As radius* : *the tangent of the given angle* :: *the side adjacent* : *the side opposite the given angle.*

(*f.*) *As sine of the angle opposite the given side* : *radius* :: *given side* : *the hypothenuse.*

4°. *When the two sides are given.*

Using the same formula as the last, we have,

(*g.*) *As the base* : *the perpendicular* :: *radius* : *the tangent of the angle at the base.*

(*h*.) *As sine the angle at the base* : *radius* :: *the perpendicular* : *the hypothenuse.*

EXAMPLES.

1. In the right-angled triangle ABC are given the hypothenuse AC=250, and the angle A=73° 44′ 23″, to solve the triangle. *Ans.* C=16° 15′ 37″.
a=240.
c=70.0003.

2. Given the angle A = 50° 30′, the hypothenuse = 142.27, to solve the triangle. *Ans.* C=39° 30′.
a=109.78.
c= 90.5.

3. Given the hypothenuse = 47.467, and one side = 37.29, to solve the triangle.
Ans. Other side=29.37.
Angles $\begin{cases} 38^\circ\ 13'\ 27''. \\ 51^\circ\ 46'\ 33''. \end{cases}$

4. Given the perpendicular=222, and the angle opposite the base=25° 15′, to solve the triangle.
Ans. 64° 15′, 104.7, and 245.45.

5. Given the base=123, and perpendicular=765, to solve the triangle. *Ans.* 80° 51′ 57″.
9° 8′ 3″.
774.82.

SOLUTION OF

OBLIQUE-ANGLED PLANE TRIANGLES.

The solution of all oblique-angled plane triangles may be included under four cases:

1°. *When one side and two angles are given.*

Since the sum of the three angles of every plane triangle is equal to 180°, therefore, if we subtract the sum of any two angles of the triangle from 180°, we shall have the third angle.

Then, from Chap. II., Theor. II., we have the following

RULE.

As sine of the angle opposite the given side : sine of the angle opposite the required side : : the given side : required side.

2°. *When two sides and an angle opposite one of them are given.*

From the same theorem as above we derive the following

RULE.

As the side opposite the given angle : the other given side : : sine of the given angle : the sine of the angle opposite the other given side.

Having found a second angle, the third is obtained as above, and the side is found by Case 1°.

3°. *When two sides and their included angle are given.*

First, subtract the included angle from 180° for the *sum* of the other two angles. Then, by Theor. III., we have the following

RULE.

As the sum of the two given sides : their difference : : tangent of half the sum of the opposite angles : tangent half their difference.

Then *to half the sum of the two angles add half their dif-*

ference for the greater angle, and from half their sum take half their difference for the less angle.

The third side is found by Case 1°.

4°. *When the three sides are given.*

The student may use any of the formulas under Theorem V., from 50 to 60, for the solution of the problem; but perhaps from Theorem IV. we may have a more convenient

RULE.

Consider the longest side the base. Then,

As the base : the sum of the other two sides :: the difference of those sides : the difference of the segments made by a perpendicular let fall from the vertical angle.

Then, *to half the base add half the difference of the segments for the greater segment, and from half the base take half the difference of the segments for the less segment.* Now,

As the greater of the other two given sides of the triangle : the greater segment :: radius : the cosine of the less angle; and,

As the less of the two given sides : the less segment :: radius : the cosine of the greater angle at the base.

EXAMPLES.

1. In the triangle ABC are given the side $c=93.37$, the angle $A=30° 20'$, and the angle $C=99° 30'$, to find the other parts. *Ans.* $a=47.81$.
$b=72.697$.
$B=50° 10'$.

2. In the triangle ABC are given the side $c=364$, the angle $A=57° 15'$, and the angle $B=35° 30'$, to find the other parts. *Ans.* $a=306.49$.
$b=211.62$.
$C=87° 15'$.

3. In the triangle ABC are given the side $a=51.234$, the side $b=42.356$, and the angle $A=55°$, to find the other parts. *Ans.* $c=61.992$.
$B=42° 37' 33''$.
$C=82° 22' 27''$.

4. In the triangle ABC are given the side $b=50.24$, the side $c=43.25$, and their included angle $A=40° 15'$, to find the other parts.

Ans. $B=81° 24' 25''$, $C=58° 20' 35''$, and $a=32.829$.

5. Given the side $a=133$, the side $c=176$, and their included angle $B=73°$, to find the remaining parts.

Ans. $A=42° 50' 57''$.
$C=64° 9' 3''$.
$b=187.022$.

6. In the triangle ABC are given the side $a=50.25$, the side $b=60.5$, and the side $c=68.4$, to find the angles.

Ans. $A=45° 22' 35''$.
$B=58° 58' 18''$.
$C=75° 39' 7''$.

7. Given $a=150$, $b=140$, and $c=130$, to find the angles.

Ans. $A=67° 22' 48''$.
$B=59° 29' 23''$.
$C=53° 7' 49''$.

8. Given $a=.8706$, $b=.0916$, and $c=.7902$, to find the angles.

Ans. $A=149° 49'$.
$B=3° 1' 56''$.
$C=27° 9' 4''$.

9. Given $a=31.23879$, $b=49.00117$, and $A=32° 18'$, to find the other parts.

Ans. $B=56° 56' 56''$.
$C=90° 45' 4''$.
$c=58.456$.

Or,

$B=123° 3' 4''$.
$C=24° 38' 56''$.
$c=24.381$.

MENSURATION.

THE term *mensuration* is frequently applied to a system of rules and methods by which numerical measures of geometrical quantities are obtained. It is to this limited view of the subject that the present chapter shall be confined.

In all practical applications of mathematics, it is necessary to express magnitudes of every kind by numbers. For this purpose, a line of some determined length, as *one inch*, *one foot*, etc., is assumed as the measuring unit of lines, and the number expressing how often this unit is contained in any line is the numerical value or measure of that line.

A surface of some determinate figure and magnitude is assumed as the *measuring unit* of surfaces, and the number of units contained in any surface is the numerical measure of that surface, and is called its *area*. It is usual to assume, as the measuring unit of surfaces, a square, whose side is the measuring unit of lines.

A solid of a determinate figure and magnitude is, in like manner, assumed as the *measuring unit* of solids, and the number of units contained in any solid is its *solidity* or *volume*. The unit of solids is a cube, each of whose edges is the measuring unit of lines, and, consequently, each of its faces the measuring unit of surfaces. The Tables of Square and Cubic Measure may be found in Arithmetic (Art. 126, 127).

MENSURATION OF SURFACES.

PROBLEM I.

To find the area of a parallelogram, square, or rectangle.

From (Geom., B. IV., Prop. 3, Cor. 2), we have the following

RULE.

Multiply the base by the altitude, and the product will be the area.

EXAMPLES.

1. Find the area of a parallelogram, the base being 15 chains and the altitude 10.50 chains.
Ans. 15 acres, 3 roods.

2. Find the area of a square each side of which is 80 chains. *Ans.* 640 acres.

3. Find the area of a rectangle whose base is 100 feet and altitude 75 feet. *Ans.* 7500 square feet.

PROBLEM II.

To find the area of a triangle whose base and altitude are given.

From (Geom., B. IV., Prop. 3, Cor. 3), we have the following

RULE.

Multiply the base and altitude together, then half the product will be the area.

EXAMPLES.

1. Find the area of a triangle whose base is 5 chains and altitude 3.50 chains. *Ans.* 3 roods, 20 perches.

2. What is the area of a triangle whose base is 40 rods and altitude 8 rods? *Ans.* 1 acre.

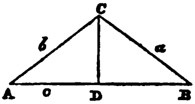
PROBLEM III.

To find the area of a triangle when its angles and one side are given.

Solution. By (prob. 2), the area of the triangle

$$\mathrm{ABC} = \tfrac{1}{2}\mathrm{CD} \times \mathrm{AB} \quad . \quad . \quad . \quad (1)$$

$$\therefore \text{Area of ABC} = \frac{c \times \mathrm{CD}}{2}.$$



But, by (th. 2, Trig.), $c = \dfrac{a \sin. \mathrm{C}}{\sin. \mathrm{A}}$, and (th. 1) $\mathrm{CD} = a \sin. \mathrm{B}$.

Substitute these in (equation 1), we have

$$\text{Area of ABC} = \frac{a^2 \sin. \mathrm{B} \sin. \mathrm{C}}{2 \sin. \mathrm{A}};$$

or $$2, \text{ Area of ABC} = \frac{a^2 \text{ sin. B sin. C}}{\text{sin. A}}.$$

Or, in logarithms,

Log. *double area* $= 2$ log. $a +$ log. sin. B $+$ log. sin. C $+$ ar. co. log. sin. A $- 20$.

From which we derive the following

RULE.

To twice the logarithm of the given side add the logarithmic sines of the adjacent angles, and the arithmetical complement o the logarithmic sine of the opposite angle, and reject 20, and th sum will be the logarithm of double the area of the triangle.

EXAMPLES.

1. Given one side of a triangle equal 24.32 chains, and the angles adjacent 63° and 74°, to find the area.

Solution.

$a = 24.32$ twice log.	2.771928
$B = 63°$ sin.	9.949881
$C = 74°$ sin.	9.982842
$A = 43°$ ar. co. sin.	0.166217
Double area $= 742.793$ sq. chs.	2.870868

$\therefore$ Area $= 37$ acres, 22 perches.

2. Given a side equal 19 chains, and the adjacent angles 60° and 50°, to find the area.

Ans. 12 acres, 2 roods, 39 perches.

PROBLEM IV.

To find the area of a triangle when two sides and their included angle are given.

Solution. Area of ABC $= \frac{1}{2}c \times \text{CD}$.

But, by (th. 1, Plane Trig.), $\text{DC} = a$ sin. B;

$$\therefore \text{Area of ABC} = \tfrac{1}{2}ac \text{ sin. B}$$

$$2 \text{ Area ABC} = ac \text{ sin. B} \quad . \quad . \quad . \quad (1)$$

Or, in logarithms,

$$\text{log. double area} = \text{log. } a + \text{log. } c + \text{log. sin. B} \quad . \quad (2)$$

Cor. *The area of a parellelogram is equal to ac sin.* B (3)

From (1) we perceive that *the double area of a triangle is equal to the product of the two sides into the natural sine of the included angle.* And from (2) we obtain the following

RULE.

Add together the logarithms of the two sides and the logarithmic sine of the included angle, reject 10, *and the remainder will be the logarithm of double the area of the triangle.*

EXAMPLES.

1. Given the side $b=15.36$ chains, the side $c=11.46$ chains, and the included angle $A=47° 30'$, to find the area.

Here	$b=15.36$ log.	1.186391
	$c=11.46$	1.059185
	$A=47° 30'$ sin.	9.867631
Double area=129.78 sq. chs.		2.113207

$\therefore$ Area=6 acres, 1 rood, 38 perches.

2. Given the side $a=18.23$ chains, the side $b=13.84$ chains, and their included angle $C=66° 30'$, to find the area. *Ans.* 11 acres, 2 roods, 11 perches.

PROBLEM V.

To find the area of a triangle when the three sides are given.

From (prob. 4),

$$\text{Area of } ABC=\tfrac{1}{2}ac \sin. B.$$

But, by (equation 51, Plane Trig.),

$$\sin. B=\frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)};$$

$$\therefore \text{Area of } ABC=\tfrac{1}{2}ac \sin. B=\sqrt{s(s-a)(s-b)(s-c)}.$$

Or, in logarithms,

$$\text{Log. area } ABC=\tfrac{1}{2}\{\log. s+\log. (s-a)+\log. (s-b)+\log. (s-c)\}.$$

In which s denotes the half sum of the three sides of the triangle. From which we derive the following

RULES.

From half the sum of the three sides subtract each side separately; then multiply the half sum and the three remainders together, and the square root of their continued product will be the area of the triangle. Or,

RULE II.

Add together the logarithms of the half sum and the three re-

mainders and half the sum of these logarithms will be the logarithm of the area.

EXAMPLES.

1. Given the three sides of a triangular field equal to 21.28, 22.56, and 18 chains respectively, to find the area.

Here $s=30.92$; $s-a=9.64$; $s-b=8.36$; $s-c=12.92$; $\therefore$ by Rule I.,

$$\sqrt{30.92\times9.64\times8.36\times12.92}=179.429 \text{ sq. chs.}$$

Or, by Logarithms, Rule II.,

$$\begin{array}{rr} s=30.92 \text{ log.} & 1.490239 \\ s-a=9.64 & 0.984077 \\ s-b=8.36 & 0.922206 \\ s-c=12.92 & \underline{1.111263} \\ & 2)\overline{4.507785} \\ \text{log. area}= & \overline{2.253892} \end{array}$$

$\therefore$ Area $=179.429$ sq. chs. $=$
17 acres, 3 roods, 30.8 perches.

2. Given the three sides of a triangular field equal to 49, 50.25, and 25.69 chains respectively, to find the area.

Ans. 61 acres, 2 roods nearly.

PROBLEM VI.

To find the area of a trapezoid.

RULE.

Multiply half the sum of the two parallel sides by the altitude of the trapezoid. (Geom., B. I., th. 29.)

Or, *Multiply the altitude by the distance between the middle points of its inclined sides.*

EXAMPLES.

1. What is the area of a trapezoid the parallel sides of which are 8 and 12 chains respectively, and the perpendicular distance between them 5 chains?

$$5\left(\frac{12+8}{2}\right)=50 \text{ sq. chs.}=5 \text{ acres.}$$

2. What is the area of a trapezoid the parallel sides of which are 12.25 and 7.50 chains, and its altitude 15.40 chains?

Ans. 15 acres, 33 perches.

The area of a trapezium or of any irregular polygon may be found by drawing diagonals, and thus dividing it into triangles.

PROBLEM VII.

To find the area of a regular polygon.

From (Geom., B. IV., th. 16), we derive the following

RULE.

Multiply half the perimeter of the polygon by the perpendicular let fall from the centre on one of its sides, and the product will be the area.

If the perpendicular be not given, it may be obtained in the following manner:

Divide 360 degrees into twice as many equal parts as there are sides in the polygon; then each one of these parts will be equal to half the angle subtended by a side.

Now, half of one of the sides of the polygon, the perpendicular, and the line drawn from the centre to the extremity of a side, constitute the base, perpendicular, and hypothenuse of a right-angled triangle, of which we have the base equal to half the side and the angles, from which the perpendicular can be determined.

EXAMPLE.

Find the area of a regular pentagon the side of which is 25 feet.

Here $\frac{360}{10}=36°$ = the vertical angle of the right-angled triangle.

Then (Prop. 1, Plane Trig.),

$$\tan. A=\frac{p}{b};$$

or $p=b \tan. A$, in which p = perpendicular, and b = half side = 12.5 feet; and $A=90°-36°=54°$.

Putting it into logarithms,

$$\log. p=\log. b+\log. \tan. A-10.$$

$$\begin{array}{ll} b=12.5 \text{ log.} & 1.096910 \\ A=54° \text{ tan.} & \underline{10.138739} \\ p=17.205 & 1.235649 \end{array}$$

$\therefore \frac{1}{2}(125\times 17.205)=1075.3125$ sq. feet, *Ans.*

If we take the side of a regular polygon equal unity, then, by the above formulas, we shall find the area of a

Triangle, equilateral,	3	sides,	0.433012
Square,	4	"	1.000000
Pentagon,	5	"	1.720477
Hexagon,	6	"	2.598076
Heptagon,	7	"	3.633912
Octagon,	8	"	4.828427
Nonagon,	9	"	6.181824
Decagon,	10	"	7.694208
Undecagon	11	"	9.365639
Dodecagon	12	"	11.196152

Now the areas of similar polygons are to each other as the squares of their like sides (Geom., B. IV., th. 11). Hence, to find the area of any regular polygon, we shall have the following

RULE.

Multiply the square of the side of the polygon by the tabular area set opposite the polygon of the same number of sides.

Thus:

Ex. Find the area of a regular decagon whose side is 10 feet.

$$\text{Tab. area} = 7.694208, \text{ and } 10^2 = 100;$$

$$\therefore 7.694208 \times 100 = 769.4208 \text{ sq. feet, } \textit{Ans.}$$

PROBLEM VIII.

Given the diameter of a circle to find the circumference; or the circumference to find the diameter.

From (Geom., B. IV., th. 17), we have the following:

$$\text{cir.} = 2\pi r$$

$$\therefore \quad r = \frac{\text{cir}}{2\pi}, \text{ where } \pi = 3.1416 \text{ nearly};$$

$$\therefore \quad 2r = \frac{\text{cir}}{\pi} = \text{diameter}.$$

Hence, *multiply the diameter by* 3.1416, *and the product will be the circumference;* or, *divide the circumference by* 3.1416, *and the quotient will be the diameter.*

PROBLEM IX.

To find the length of an arc of a circle containing any number of degrees.

From (Plane Trig., Chap. I., sec. 4), we derive the following

RULE.

Multiply the number of degrees in the given arc by the radius of the circle, and divide the product by 57.29578.

If the arc contains degrees and minutes, reduce the minutes to the decimal of a degree.

EXAMPLES.

1. Required the length of the arc whose radius is 9, and the number of degrees 38° 56′ 6″. *Ans.* 6.11701.

2. Required the length of an arc of 60°, the radius being 10.

PROBLEM X.

To find the area of a circle.

From (Geom., B. IV., th. 17), we have the following

RULE.

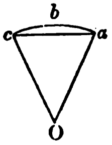
Square the radius, and multiply by 3.1416;
Or, *Square the diameter, and multiply by* .7854.

EXAMPLE.

What is the area of a circle whose diameter is 7?
Ans. 38.4846.

PROBLEM XI.

To find the area of a sector of a circle.



Let $abcO$ be any sector of a circle.
Put $Oa = Oc = r$;
And the angle $aOc = \theta$.
Then the arc $abc = r\theta \div \frac{180°}{\pi}$ (Trig., Chap. I.).

Now, if the sector were a quadrant, the arc abc would be equal to $\frac{1}{2}\pi r$, and the area of the quadrant would be $= \frac{1}{4}\pi r^2$.

But, in equal circles, sectors are to each other as the arcs on which they stand.

$$\therefore \text{Area of the sector} : \text{area of quadrant} :: \frac{\pi r\theta}{180^\circ} : \tfrac{1}{2}\pi r;$$

$$\text{or,} \qquad \text{Area of the sector} : \tfrac{1}{4}\pi r^2 :: \frac{\pi r\theta}{180^\circ} : \tfrac{1}{2}\pi r.$$

$$\therefore \text{Area of sector} = \tfrac{1}{2}r \times \frac{\pi r\theta}{180^\circ}.$$

From which we derive the following

RULE.

Multiply half the radius by the arc on which the sector stands.

EXAMPLES.

1. Find the area of a sector of a circle whose arc contains 18°, the diameter being 3 feet.

$$\text{Area} = \tfrac{1}{2}r \times \frac{\pi r\theta}{180^\circ} = .75 \times \frac{3.14159 \times 1.5 \times 18}{180} =$$

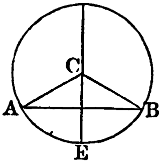
$$.353428875 \text{ sq. ft.}$$

2. Find the area of a sector of a circle whose arc contains 10°, the diameter being 10 feet.

Ans. 2.1816 sq. ft.

PROBLEM XII.

To find the area of a segment of a circle.



The area of the sector CAEB (by the last problem) is equal to $\frac{1}{2}r \times \frac{\pi r\theta}{180^\circ}$;

And the area of the triangle CAB= $\frac{1}{2}r^2$ sin. θ (by prob. 1).

Therefore the area of the segment AEB, being the difference between the sector and the triangle, is $\left(\frac{\pi\theta}{360^\circ} - \frac{1}{2}\text{ sin. }\theta\right)r^2$;

And the area of the segment AGB, which is greater than a semicircle, is the sum of the sector AGB and the triangle ACB.

Hence, to determine the area of a segment of a circle, we have the following

RULE.

Find the area of a sector which has the same arc as the segment; also the area of the triangle formed by the chord of the segment, and the radii of the sector.

Then take the difference of these areas when the segment is less than a semicircle, and the sum when it is greater.

EXAMPLES.

1. Find the area of a segment of a circle, the chord being 12 feet, and radius 10 feet. *Ans.* 16.3504 sq. ft.

2. Find the area of a segment of a circle whose arc contains 10°, the radius being 10 feet. *Ans.* 6.336 sq. ft.

PROBLEM XIII.

To find the area of a circular ring; or the area included between the circumferences of two circles which have a common centre.

Let R denote the radius of the greater circle.

Let r denote the radius of the less circle.

Then (Geom., B. IV., th. 17) the area of the larger is πR^2, and the area of the less is πr^2.

$$\therefore \pi R^2 - \pi r^2 = \text{area of the circular ring};$$

or, $$\pi(R^2 - r^2), \text{ or } \pi(R+r)(R-r).$$

Whence we derive the following

RULE.

Multiply the sum of the two radii by their difference, and that product by 3.1416.

EXAMPLE.

What is the area of the space between two concentric circles whose radii are 10 and 6? *Ans.* 201.0624.

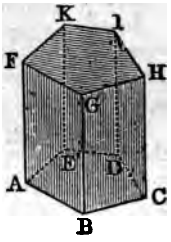
MENSURATION OF SOLIDS.

THE mensuration of solids consists of two parts:
1°. The mensuration of their surfaces;
2°. The mensuration of their volumes.

SURFACES OF SOLIDS.

PROBLEM I.

To find the convex surface of a right prism.



Let ABCDE—K be a right prism.

Then, since the planes FE, FB are perpendicular to the base, their line of intersection FA is also perpendicular to the base (Geom., B. V., Prop. XVII.), and therefore the angle FAB is a right angle. Hence the area of ABGF is $=AB \times AF$. Similarly for all the other parallelograms. Hence the area of the parallelograms taken together is

$$AF \times (AB+BC+CD+DE+EA).$$

From which we derive the following

RULE.

Multiply the perimeter of the base by the altitude, and the product will be the convex surface.

When the entire surface is required, we must add the area of the two bases to the convex surface.

EXAMPLE.

What is the convex surface of a cube, the length of each side being 20 feet? *Ans.* 2400 sq. feet.

PROBLEM II.

To find the convex surface of a right cylinder.

Since the cylinder is the limit of the convex surface of the inscribed prism, whatever may be the number of its sides, therefore *the convex surface of a cylinder is obtained by multiplying the circumference of its base by the altitude.*

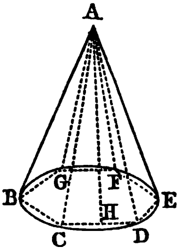
EXAMPLE.

What is the convex surface of a cylinder whose length is 20 feet, and the circumference of its base 3 feet?

Ans. 60 sq. feet.

PROBLEM III.

To find the convex surface of a right cone, or right pyramid.



Let BCDE—A be any right cone, BCDE its circular base.

In the circle inscribe any regular polygon, BCDEFG, and join AC, AD. . . . Then will these be the edges of the inscribed pyramid.

Bisect any side, CD in H; join AH; then AH is perpendicular to CD (Geom., B. I., th. 3, Cor. 1); and therefore the area of the triangle $ACD = \frac{1}{2}CD \times AH$.

Now the line joining A with the point of bisection of any other side of the polygon is equal to AH.

Hence *the convex surface of the inscribed pyramid is*

$$= \frac{1}{2}AH\ (BC + CD + DE +, \&c.),$$
$$= \frac{1}{2}AH \times \text{perimeter of the base.}$$

Now this is true whatever be the number of sides of the polygon forming the base, and, hence, is true in the limit. If, therefore, we increase the number of the sides indefinitely, the polygon will become the circumference of the circle, and AH the slant height of the cone. Therefore,

The convex surface of a right pyramid or a right cone is equal to half the slant height multiplied by the perimeter, or circumference of its base.

Rem. If we suppose the surface of the cone to be unwrapped, it is evident that its surface will be a sector of a circle whose radius is the slant height of the cone, and the arc of the same length as the circumference of the base of the cone. It is plain that the area of this sector is, as it should be, the same as that of the convex surface of the cone.

PROBLEM IV.

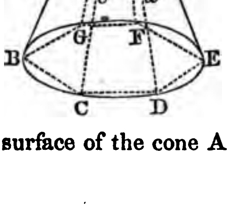
To find the convex surface of a frustum of a right pyramid.

Since the section is similar to the base (Geom., B. VII., Prop. VII.), and the base is a regular polygon, it follows that the sides of the section are all equal to each other. Hence the convex surface is composed of equal trapezoids, whose common altitude is equal to the slant height of the frustum. But the area of any one of these trapezoids is equal to half the sum of the parallel sides multiplied by the altitude (Prob. VI.). Hence *the convex surface of a frustum of a right pyramid is obtained by multiplying half the sum of the perimeters of the two bases by the slant height of the frustum.*

EXAMPLE.

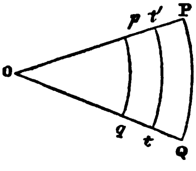
How many square feet are in the surface of a frustum of a square pyramid whose slant height is 10 feet; also each side of the greater base is 3 feet 4 inches, and each side of the less base is 2 feet 2 inches? *Ans.* 110 sq. feet.

PROBLEM V.



To find the convex surface of the frustum of a right cone.

Let *b*BE*c* be a frustum of a right cone, and suppose a sector, OPQ, of a circle to be described with a radius OP = AB; and if its base PQ=circumference BE, we have seen that the area of the sector OPQ is the same as the convex surface of the cone ABE (Prob. III., Rem.).



Take $Op=Ab$, and describe the arc pq; then, as before, the area of Opq=convex surface of the cone Abc; and, therefore, the convex surface of the cone is equal to $pqQP$.

Let the angle $POQ=\theta$.

$$OP=OQ=R.$$
$$Op=Oq=r.$$

Then the area of $POQ=\frac{1}{2}R^2\times\frac{\pi\theta}{180^\circ}$, and

$$pOq=\tfrac{1}{2}r^2\times\frac{\pi\theta}{180^\circ}.$$

$$\therefore \text{ the area of } qQPp=\tfrac{1}{2}\left(\frac{\pi R\theta}{180^\circ}+\frac{\pi r\theta}{180^\circ}\right)(R-r);$$

but $\frac{\pi R\theta}{180^\circ}$ is equal to the circumference of the lower base; $\frac{\pi r\theta}{180^\circ}$=circumference of the upper base (Prob. XI., Mens. Surfaces); and $R-r$=slant height of the frustum. Therefore, to find the convex surface of a frustum of a cone, we have the following

RULE.

Multiply half the sum of the circumferences of the two bases by the slant height.

Cor. If we bisect pP in t', and describe the arc $t't$, then

$$\text{arc } tt'=\tfrac{1}{2}(R+r)\frac{\pi\theta}{180^\circ}.$$

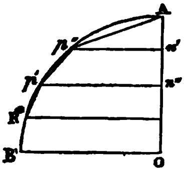
Therefore the convex surface of a frustum of a cone *is equal to the product of the slant height, and the circumference of a section at equal distances between the two bases.*

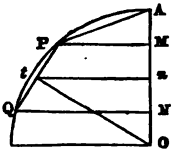
EXAMPLE.

What is the convex surface of a cone whose slant height is 10 feet, and the circumferences of its bases are 6 feet and 4 feet? *Ans.* 50 sq. feet.

PROBLEM VI.

To find the surface of a spherical zone, and also that of a sphere.





Let AB be a quadrant of a circle whose radius is OA or OB. If we suppose the quadrant to revolve around AO it will describe a hemisphere; and if we suppose a number of *equal* chords, Ap', $p'p''$, &c., to be drawn from point to point of AB, these chords, in the revolution, will describe frustums of cones. Now the arc AR is the limit of the chords $Ap' + p'p'' +$, &c., and hence the surface of the portion of the sphere described by AR will be the limit of the sum of the frustums of cones described by Ap', $p'p''$, &c.

Let PQ be one of these chords; draw PM, QN, perpendicular to AO. Draw Ot perpendicular to PQ, and tn perpendicular to AO. Now tn is the radius of the mean section of the frustum of the cone described by PQ, and therefore

the area of that frustum $= 2\pi . tn \times PQ$ (Prob. V.).

Now $tn = Ot . \sin. tOA$, and

$MN = PQ . \sin. PQN$; or, since $PQN = tOA$,

$MN = PQ . \sin. tOA$;

$\therefore tn \times PQ = Ot \times MN$.

$\therefore$ surface of the frustum $= 2\pi \times Ot \times MN$.

If we draw $p'n'$, $p''n''$, &c., perpendicular to AO, we shall have, for the sum of the frustums of the cones,

$2\pi \times Ot\ (An' + n'n'' +$, &c.)

$= 2\pi \times Ot \times AN$, if we only consider the portion of the sphere described by AR.

This is true whatever may be the number of chords, and is, therefore, true in the limit; but in the limit Ot becomes the radius of the sphere; therefore the surface of a portion of the sphere whose height is $AN = 2\pi \times OA \times AN$.

But $2\pi \times OA =$ circumference of a great circle of the sphere, and AN is the altitude of the zone. Therefore,

To find the surface of a spherical zone, we have the following

RULE.

Multiply the altitude of the zone by the circumference of a *great circle of the sphere, and the product will be the surface required.*

Again, if we take the whole sphere, then AN becomes 2AO; therefore the

$$\text{surface of a sphere} = 4\pi \times \overline{AO}^2.$$

Hence, *to find the surface of a sphere*, we have the following

RULE.

Multiply four times the square of the radius by 3.1416.

If we imagine a right cylinder to be described about a sphere, its concave surface will be equal to the circumference of a great circle multiplied by the diameter of the sphere. Hence,

The surface of a sphere is equal to the convex surface of the circumscribing cylinder.

EXAMPLE.

What is the surface of a globe whose diameter is 17 inches? *Ans.* 6.305 sq. feet.

VOLUME OF SOLIDS.

PROBLEM I.

To find the volume of a right prism or cylinder.

RULE.

Multiply the area of the base by the altitude of the prism or cylinder, the product will be the volume (Geom., B. VII., Prop. 5, Cor. 2).

EXAMPLE.

Required the volume of a triangular prism whose length is 20 feet, and the sides of its triangular bases 3, 4, and 5 feet respectively. *Ans.* 120 cubic feet.

PROBLEM II.

To find the volume of a pyramid or cone.

RULE.

Multiply the area of the base by one third of the altitude of the pyramid or cone, and the product will be the volume (Geom., B. VII., Prop. 10, Cor. 1 and 2).

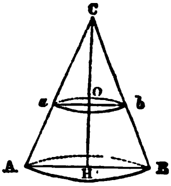
EXAMPLE.

What is the volume of a cone, the radius of its base being 9 inches, and its height 15 feet?

Ans. 8.8357 cubic feet.

PROBLEM III.

To find the volume of a frustum of a right cone.



Let $abBA$ be a frustum of the cone ABC. Join COH where O and H are the centres of the bases of the frustums.

Let $aO=r$; $AH=R$; $OH=h$, and $CO=x$;

$$\text{Then,}\quad \left.\begin{aligned}\text{vol. CAB}&=\tfrac{1}{3}\pi r^2.x\\ \text{vol. CEF}&=\tfrac{1}{3}\pi R^2(x+h)\end{aligned}\right\}\text{ by Prob. II.}$$

$$\begin{aligned}\therefore \text{vol. of frustum}&=\tfrac{1}{3}\pi\{R^2(x+h)-r^2x\}\\ &=\tfrac{1}{3}\pi.R^2h+\tfrac{1}{3}\pi.(R^2-r^2)x.\end{aligned}$$

But, by similar triangles,

$$x+h:R::x:r;$$

or,

$$h:x::R-r:r.$$

$$\therefore x=\frac{hr}{R-r}.$$

$$\begin{aligned}\text{Hence, vol. of frustum}&=\tfrac{1}{3}\pi.R^2h+\tfrac{1}{3}\pi(R^2-r^2)\frac{hr}{R-r}\\ &=\tfrac{1}{3}\pi h(R^2+Rr+r^2).\end{aligned}$$

From which we perceive *that the volume of the frustum of a cone is equal to the sum of the volumes of three cones, whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional to the two.* From the above formula we also derive the following

RULE.

To the sum of the squares of the radii of the two bases add the product of these two radii; multiply the sum by one third the altitude, and that product by 3.1416.

EXAMPLE.

What is the volume of the frustum of a cone, the diameter of the greater end being 5 feet, that of the less end 3 feet, and the altitude 9 feet? *Ans.* 115.4538 cubic feet.

To find the volume of a frustum of a regular pyramid.

Add together the areas of the two bases and a mean proportional between them; multiply this sum by one third of the altitude.

EXAMPLES.

1. What is the volume of the frustum of a cone, the diameter of the greater end being 5 feet, that of the less end 3 feet, and the altitude 9 feet?

Ans. 115.4538 cubic feet.

2. What is the volume of the frustum of a square pyramid, the larger end being 4 feet each way, the smaller end 3 feet, and the height 6 feet? *Ans.* 74 cubic feet.

PROBLEM IV.

To find the volume of a sphere.

Since every sphere is two thirds of the circumscribing cylinder (Geom., B. VII., Prop. 12), and the volume of the circumscribing cylinder

$$=\pi r^2 \times 2r \text{ (Prob. I. Vol. Solids)};$$

$$\therefore \text{vol. cir. cyl.} = 2\pi r^3.$$

Or, if we put $2r = D = \text{diameter} \therefore r = \frac{1}{2}D$, and

$$\text{vol. cir. cyl.} = \tfrac{1}{4}\pi D^3;$$

$$\therefore \text{vol. of sphere} = \tfrac{1}{6}\pi . D^3.$$

Hence we have the following

RULE.

Cube the diameter of the sphere, and multiply by $\frac{1}{6}\pi$.

$$\tfrac{1}{6}\pi = 0.5236.$$

EXAMPLE.

What is the volume of a sphere whose diameter is 6 feet? *Ans.* 113.0976 cubic feet.

PROBLEM V.

To find the volume of a spherical segment or zone.

Let a = area of a great circle of the sphere.
d = diameter.
h = altitude of zone or segment.

Then (Geom., B. VII., Prop. 12, Scholium),

$$\text{vol. of segment} = \tfrac{2}{3}\cdot\frac{a}{d^2}(3d-2h)h^2.$$

But $a = \frac{1}{4}\pi d^2$.

$$\therefore \text{vol. of segment} = \tfrac{1}{6}\pi(3d-2h)h^2.$$

From which we derive the following

RULE.

From three times the diameter of the sphere subtract twice the altitude of the segment; multiply the remainder by the square of the altitude, and this last product by 0.5236 *for the volume required.*

Rem. 1. If the segment has two bases, and both on the same side of a plane passing through the equator, we may find the volume, by the above rule, under the supposition that we have two segments; then take the difference of the volumes for the volume of the segment.

Rem. 2. If the bases lie on different sides of the equator, take the sum of the volumes from the volume of the sphere for the volume of the segment.

The formula for Rem. 1 is,

$$\text{vol. seg.} = \tfrac{1}{6}\pi\{3d(h^2-h'^2)-2(h^3-h'^3)\}.$$

And the formula for Rem. 2 is,

$$\text{vol. seg.} = \tfrac{1}{6}\pi\{d^3-3d(h^2+h'^2)-2(h^3+h'^3)\}.$$

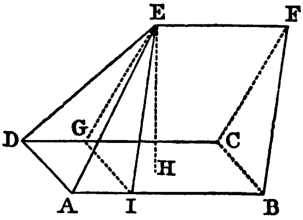
EXAMPLE.

In a sphere whose diameter is 21 inches, what is the volume of a segment whose height is 4.5 inches?

Ans. 572.5566 cubic inches.

PROBLEM VI.

To find the volume of a wedge.



Def. *A wedge* is a solid, bounded by five planes; the rectangular base, ABCD, two trapezoids, ABFE, DGCFE, intersecting in the edge EF, and two triangular ends, ADE, BCF.

The *altitude* of the wedge is the perpendicular drawn from any point in the edge to the plane of the base, as EH.

Put l=AB=length of the base.
l'=EF= " " edge.
b=BC=AD=breadth of the base.
h=HE=altitude of the wedge.

Now if the length of the base and that of the edge are equal to each other, it is plain that the wedge is half a parallelopiped of the same base and altitude.

If the length of the base is greater than that of the edge, let a plane, EGI, be drawn parallel to BCF. The wedge will then be divided into two parts, *viz.*, the triangular prism BCG—F, and the pyramid AIGD—E.

The vol. of prism$=\frac{1}{2}bhl'$ (Vol. of Solids, Prob. I.)
vol. of pyr. $=\frac{1}{3}bh(l-l')$, (Vol. of Solids, Prob. II.)
$\therefore$ vol. of wedge$=\frac{1}{6}bh(2l+l')$.

If the length of the base is less than that of the edge, then the wedge will be equal to the difference between the prism and pyramid.

$\therefore$ vol. of wedge$=\frac{1}{6}bh(2l+l')$,

the same as above.

Hence we derive the following

RULE.

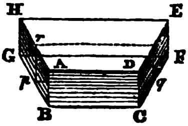
To twice the length of the base add the length of the edge, and multiply the sum by one sixth of the product of the height of the wedge and breadth of the base.

EXAMPLE.

What is the volume of a wedge whose base is 30 inches long and 5 inches broad, its altitude 12 inches, and the length of the edge 24 inches? *Ans.* 840 cubic inches.

PROBLEM VII.

To find the volume of a rectangular prismoid.



Def. *A rectangular prismoid* is a solid bounded by six planes, of which the two bases are rectangles, having their corresponding sides parallel, and the four other sides are trapezoids.

Put $l'=AD=BC$;
$b'=AB=CD$;
$l=HE=GF$;
$b=HG=EF$;
$h=$alt. of the prismoid;
$M=$the length of the mid section$=pq$;
$m=$the breadth " " $=pr$.

If a plane be passed through the opposite edges of the upper and lower bases, the prismoid will be divided into two wedges, whose bases are the bases of the prismoid, and whose edges are l' and l.

The volume of these wedges, by Prob. VI., is

$$\tfrac{1}{6}bh(2l+l')+\tfrac{1}{6}b'h(2l'+l)$$

$$=\tfrac{1}{6}h(2bl+bl'+2b'l'+b'l)=\text{vol. of the prismoid.}$$

But, since M and m are the length and breadth of the mid sections, we have

$$2M=l+l', \text{ and } 2m=b+b'$$

$$\therefore 4Mm=(l+l')(b+b')=bl+bl'+b'l+b'l'.$$

Substituting $4Mm$ for its value in the preceding expression, we have for the value of the prismoid

$$\tfrac{1}{6}h(bl+4Mm+b'l'),$$

from which we derive the following

RULE.

To the sum of the areas of the two bases add four times the area of a middle section, and multiply the sum by one sixth of the altitude.

EXAMPLE.

What is the volume of a block of marble in the form of a rectangular prismoid, the length and breadth of one end being 16 and 12 inches, and of the other 7 and 4 inches, the length of the block being 24 feet?

Ans. $16\frac{1}{3}$ cubit feet.

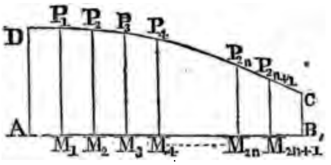
PROBLEM VIII.

To determine the volume of a railroad cutting.

In Prob. VII. the figure is very nearly that of a portion of a railroad cutting in which AC is the road, and AG and DF the sloping sides of the embankment; hence the solid contents required can be found by means of the rule given in the last problem.

It is to be observed that the rule requires that HG and EF be straight lines, or, as an approximation, to be very nearly straight lines; or AG to be a plane, which is not true if the cutting is a long one. For this reason we shall derive the following:

Let ABCD represent a section, made by a vertical plane, of the hill to be cut through; AB the level of the road; and suppose sections of the cutting to be made by planes perpendicular to AB, at *equal distances* along that line, *viz.*, at M_1, M_2 . . . . . . M_{2n+1}.



Let the terminal sections at A and B be denoted by a and b respectively, and the sections at P_1, M_1, P_2, M_2 . . . . . &c., P_{2n+1}, M_{2n+1}, be denoted by $p_1, p_2 \ldots\ldots p_{2n+1}$, *the number of sections being odd.* And let the common distance between the sections equal h. Then (by Prob. VII.),

$$\text{vol. of } AM_2P_2D = \tfrac{1}{6}AM_2(a+4p_1+p_2);\text{ or}$$

$$= \tfrac{1}{3}h(a+4p_1+p_2);\text{ and}$$

$$\text{vol. of } P_2M_2M_4P_4 = \tfrac{1}{3}h(p_2+4p_3+p_4),\text{ and so on;}$$

$$\text{vol. of } P_{2n}M_{2n}BC = \tfrac{1}{3}h(p_{2n}+4p_{2n+1})+b.$$

Hence, by addition, the whole required volume

$$=\tfrac{1}{3}h\{(a+b)+4(p_1+p_3+, \&c. \ldots\ldots p_{2n+1})+2(p_2+p_4+\ldots\ldots p_{2n})\}.$$

Whence we derive the following

RULE.

Between the first and last sections, make an odd number of sections at equal distances along the road, the planes of the sections being perpendicular to the road; then one third part of the common distance, multiplied by the sum of the first and last sections, with four times the sum of the odd sections and twice the sum of the even sections, gives the volume required.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

1. SPHERICAL TRIGONOMETRY treats of the relations subsisting between the parts of a solid angle formed by the inclination of three planes; or, supposing the angular point the centre of a sphere, the relations between the sides and angles of the triangular surface of the sphere inclosed by the three planes.

2. A *spherical triangle* is the portion of the surface of a sphere inclosed by the three planes which form the solid angle at the centre: it is bounded by three arcs of great circles.

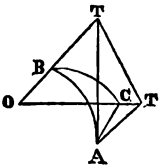
3. A *spherical angle* is the angle contained by any two of the planes passing through the centre of the sphere.

4. The *sides* of a spherical triangle are the three arcs which measure the rectilinear angles at the centre respectively. The sense in which the spherical triangle is employed being once understood, we may transfer our attention from the solid angle to the triangle, and the solid angle need not be represented in our diagrams.

The first formula which we will establish is to spherical trigonometry what the expression for the sine of two angles is to plane trigonometry; that is, it serves as a key to all the rest.

PROBLEM I.

To express the cosine of an angle of a spherical triangle in terms of the sines and cosines of the sides.



Let ABC be a spherical triangle, O the centre of the sphere. Let the angles of the triangle be denoted by the large letters A, B, C, and the sides opposite to them by the corresponding small letters a, b, c. At the point A draw AT, a tangent to the arc AB; and AT′ a tangent to the arc AC.

Then the spherical angle A is equal to the angle TAT′ between the tangents, because the tangents AT, AT′ are drawn perpendicularly to the radius of the sphere at the point A. Therefore the angle TAT′ is equal to the angle made by two planes passing through AB, AC; that is, to the spherical angle TAT′.

Join OB, and produce it to meet AT in T.

Join OC, and produce it to meet AT in T′; join TT′;

then

$$\frac{OT}{OC} = \text{sec. AB} = \text{sec. } c;$$

$$\frac{OT'}{OC} = \text{sec. AC} = \text{sec. } b;$$

$$\frac{AT}{OC} = \text{tan. AB} = \text{tan. } c;$$

$$\frac{AT'}{OC} = \text{tan. AC} = \text{tan. } b.$$

Then, in the triangle TOT′, we have

$$\overline{TT'}^2 = \overline{OT}^2 + \overline{OT'}^2 - 2OT \,.\, OT' \cos. TOT'.$$

Divide each member of the equation by $\overline{OC}^2$, we have

$$\frac{\overline{TT'}^2}{OC^2} = \frac{\overline{OT}^2}{OC^2} + \frac{\overline{OT'}^2}{OC^2} - 2\frac{OT}{OC} \cdot \frac{OT'}{OC} \cos. TOT'$$

$$= \text{sec.}^2 c + \text{sec.}^2 b - 2 \text{ sec. } c \text{ sec. } b \cos. a \quad . \quad (a)$$

Again, in the triangle TAT′, we have

$$\overline{TT'}^2 = \overline{AT}^2 + \overline{AT'}^2 - 2AT \,.\, AT' \cos. TAT'.$$

Divide each member of this equation by OC^2, it becomes

$$\frac{\overline{TT'}^2}{OC^2} = \frac{\overline{AT}^2}{OC^2} + \frac{\overline{AT'}^2}{OC^2} - 2\frac{AT}{OC} \cdot \frac{AT'}{OC} \cos. TAT'$$

$$= \text{tan.}^2 c + \text{tan.}^2 b - 2 \text{ tan. } c \text{ tan. } b \cos. A \quad (b)$$

Equating (*a*) and (*b*), we have

$$\text{tan.}^2 c + \text{tan.}^2 b - 2 \text{ tan. } c \text{ tan. } b \cos. A = \text{sec.}^2 c + \text{sec.}^2 b - 2 \text{ sec. } c \text{ sec. } b \cos. a;$$

$$\text{sec.}^2 c - \text{tan.}^2 c + \text{sec.}^2 b - \text{tan.}^2 b = 2 \text{ sec. } b \text{ sec. } c \cos. a - 2 \text{ tan. } b \text{ tan. } c \cos. A.$$

But $\quad \text{sec.}^2 c - \text{tan.}^2 c = 1$, and $\text{sec.}^2 b - \text{tan.}^2 b = 1$.

Substituting these in the above equation, it becomes

$$2 = 2 \text{ sec. } b \text{ sec. } c \cos. a - 2 \text{ tan. } b \text{ tan. } c \cos. A;$$

or $\quad \text{tan. } b \text{ tan. } c \cos. A = \text{sec. } b \text{ sec. } c \cos. a - 1.$

Hence $$\frac{\sin. b \sin. c \cos. A}{\cos. b \cos. c}=\frac{\cos. a}{\cos. b \cos. c}-1$$

$$=\frac{\cos. a-\cos. b \cos. c}{\cos. b \cos. c};$$

therefore, $$\cos. A=\frac{\cos. a-\cos. b \cos. c}{\sin. b \sin. c} \quad . \quad . \quad . \quad (1)$$

In like manner we find

$$\cos. B=\frac{\cos. b-\cos. a \cos. c}{\sin. a \sin. c} \quad . \quad . \quad . \quad (2)$$

$$\cos. C=\frac{\cos. c-\cos. a \cos. b}{\sin. a \sin. b} \quad . \quad . \quad . \quad (3)$$

PROBLEM II.

Given two sides and the included angle of a spherical triangle, to determine the third side, independently of the unknown angles.

Let b, c, A be given, and let it be required to find a, independently of the angles B and C.

Solution. Resuming formula (1),

$$\cos. A=\frac{\cos. a-\cos. b \cos. c}{\sin. b \sin. c},$$

we have $\cos. a=\cos. A \sin. b \sin. c+\cos. b \cos. c.$

In order to adapt this equation to logarithmic computation, let us add and subtract $\sin. b \sin. c$ to the second member of the equation; then

$$\cos. a=\cos. A \sin. b \sin. c+\cos. b \cos. c+\sin. b \sin. c-\sin. b \sin. c;$$

$$=\cos. b \cos. c+\sin. b \sin. c+\sin. b \sin. c \cos. A-\sin. b \sin. c;$$

$$=\cos. (b-c)-\sin. b \sin. c \text{ versin. } A.$$

And by subtracting each member of this equation from unity, we have

$$1-\cos. a=1-\cos. (b-c)+\sin. b \sin. c \text{ versin. } A;$$

$$\therefore \text{versin. } a=\text{versin. } (b-c)+\sin. b \sin. c \text{ versin. } A;$$

$$=\text{versin. } (b-c)\left\{1+\frac{\sin. b \sin. c \text{ versin. } A}{\text{versin. } (b-c)}\right\}.$$

Put $$\tan.^2 \theta=\frac{\sin. b \sin. c \text{ versin. } A}{\text{versin. } (b-c)};$$

then $$\text{versin. } a=\text{versin. } (b-c)(1+\tan.^2 \theta);$$
$$=\text{versin. } (b-c) \sec.^2 \theta \quad . \quad . \quad . \quad (4)$$
Whence a is determined.

PROBLEM III.

Given two sides and the angle opposite to one of them, to determine the third side, independently of the angle opposite to the other side.

Solution. Let a, b, A be given, to find c independently of the angle B.

Equation (1): $$\cos. A=\frac{\cos. a-\cos. b \cos. c}{\sin. b \sin. c};$$

$$\therefore \sin. c \sin. b \cos. A=\cos. a-\cos. b \cos. c;$$

or $$\sin. c \tan. b \cos. A=\frac{\cos. a}{\cos. b}-\cos. c.$$

Put $$\tan. \theta=\frac{\sin. \theta}{\cos. \theta}=\tan. b \cos. A;$$

then $$\sin. c \frac{\sin. \theta}{\cos. \theta}+\cos. c=\frac{\cos. a}{\cos. b};$$

or $$\sin. c \sin. \theta+\cos. c \cos. \theta=\frac{\cos. a \cos. \theta}{\cos. b};$$

$$\therefore \cos. (c-\theta)=\frac{\cos. a \cos. \theta}{\cos. b} \quad . \quad . \quad . \quad (5)$$

Whence c may be found, θ being previously determined from the equation $\tan. \theta=\tan. b \cos. A$.

PROBLEM IV.

To express the cosine of a side of a spherical triangle in terms of the sines and cosines of the angles.

Solution. Let A, B, C, a, b, c be the angles and sides of a spherical triangle; A', B', C', a', b', c' the corresponding quantities in the polar triangle; then, by (1),

$$\cos. A'=\frac{\cos. a'-\cos. b' \cos. c'}{\sin. b' \sin. c'}.$$

But (by Spherical Geometry, Prop. 6)

$$A'=(180^\circ-a),\ a'=(180^\circ-A),\ b'=(180^\circ-B),\ c'=(180^\circ-C).$$

$$\therefore \cos. (180^\circ-a)=\frac{\cos. (180^\circ-A)-\cos. (180^\circ-B) \cos. (180^\circ-C)}{\sin. (180^\circ-B) \sin. (180^\circ-C)};$$

$$\therefore \cos. a = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C} \quad . \quad . \quad . \quad (6)$$

In like manner we obtain

$$\cos. b = \frac{\cos. B + \cos. A \cos. C}{\sin. A \sin. C} \quad . \quad . \quad . \quad (7)$$

$$\cos. c = \frac{\cos. C + \cos. A \cos. B}{\sin. A \sin B} \quad . \quad . \quad . \quad (8)$$

PROBLEM V.

Given two angles and the included side, to find the remaining angle, independently of the unknown sides.

Solution. Let A, B, c be given, to find C independently of a and b.

By (8), $$\cos. c = \frac{\cos. C + \cos. A \cos. B}{\sin. A \sin. B};$$

$$\therefore \cos. C = \cos. c \sin A \sin. B - \cos. A \cos. B.$$

Subtracting each member of this equation from unity, we have

$$1 - \cos. C = 1 - \sin. A \sin. B\ (1 - \text{versin.}\ c) + \cos. A \cos. B;$$

$$= 1 + \cos. (A+B) + \sin. A \sin. B \text{ versin. } c; \text{ or}$$

$$2 \sin.^2 \tfrac{1}{2}C = 2 \cos.^2 \tfrac{1}{2}(A+B) + \sin. A \sin. B \text{ versin. } c;$$

$$= 2 \cos.^2 \tfrac{1}{2}(A+B) \cdot \left\{1 + \frac{\sin. A \sin. B \text{ versin. } c}{2 \cos.^2 \tfrac{1}{2}(A+B)}\right\}.$$

Let $$\tan.^2 \theta = \frac{\sin. A \sin. B \text{ versin. } c}{2 \cos.^2 \tfrac{1}{2}(A+B)};$$

then $$\sin.^2 \tfrac{1}{2}C = \cos.^2 \tfrac{1}{2}(A+B) \sec.^2 \theta \quad . \quad . \quad . \quad (9)$$

PROBLEM VI.

Given two angles and the side opposite to one of them, to find the remaining angle.

Let A, B, a be given to find C.

Solution.

By (6), $$\cos. a = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C};$$

$$\therefore \sin. B \sin. C \cos. a = \cos. A \cos. B \cos. C;$$

or $$\sin. C \tan. B \cos. a = \frac{\cos. A}{\cos. B} + \cos. C;$$

or $$\sin. C \tan. B \cos. a - \cos. C = \frac{\cos. A}{\cos. B}.$$

Put $$\tan. B \cos. a = \tan. \theta = \frac{\sin. \theta}{\cos. \theta};$$

then $$\sin. C \frac{\sin. \theta}{\cos. \theta} - \cos. C = \frac{\cos. A}{\cos. B};$$

or $$\sin. C \sin. \theta - \cos. C \cos. \theta = \frac{\cos. A}{\cos. B} \cos. \theta;$$

therefore $$-\cos. (C+\theta) = \frac{\cos. A \cos. \theta}{\cos. B} \quad . \quad . \quad . \quad (10)$$

Whence C may be found, θ being known from the equation $\tan. \theta = \tan. B \cos. a$.

PROBLEM VII.

To express the sine of an angle of a spherical triangle in terms of the sines of the sides of the triangle.

By (1), $$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}.$$

By adding unity to both members of the equation, we have

$$1 + \cos. A = \frac{\cos. a - \cos. b \cos. c + \sin. b \sin. c}{\sin. b \sin. c};$$

$$= \frac{\cos. a - \cos. (b+c)}{\sin. b \sin. c};$$

$$= \frac{2 \sin. \frac{1}{2}(a+b+c) \sin. \frac{1}{2}(b+c-a)}{\sin. b \sin. c}.$$

Put $s = \frac{1}{2}(a+b+c)$; then $s-a = \frac{1}{2}(b+c-a)$; $s-b = \frac{1}{2}(a+c-b)$; $s-c = \frac{1}{2}(a+b-c)$;

$$\therefore 1 + \cos. A = \frac{2 \sin. s \sin. (s-a)}{\sin. b \sin. c} \quad . \quad . \quad (a)$$

Again, resuming the expression for cos. A, and subtracting each member from unity, we shall have

$$1 - \cos. A = \frac{\cos. b \cos. c + \sin. b \sin. c - \cos. a}{\sin. b \sin. c};$$

$$= \frac{\cos. (b-c) - \cos. a}{\sin. b \sin. c};$$

$$= \frac{2 \sin. \frac{1}{2}(a+b-c) \sin. \frac{1}{2}(a+c-b)}{\sin. b \sin. c};$$

$$= \frac{2 \sin. (s-c) \sin. (s-b)}{\sin. b \sin. c} \quad . \quad . \quad . \quad . \quad (b)$$

Multiplying equations (a) and (b), we have

$$1-\cos.^2 A=\frac{4 \sin. s \sin.(s-a) \sin.(s-b) \sin.(s-c)}{\sin.^2 b \sin.^2 c}; \therefore$$

$$\sin. A=\frac{2}{\sin. b \sin. c}\sqrt{\sin. s \sin.(s-a) \sin.(s-b) \sin.(s-c)} \quad (11)$$

In like manner,

$$\sin. B=\frac{2}{\sin. a \sin. c}\sqrt{\sin. s \sin.(s-a) \sin.(s-b) \sin.(s-c)} \quad (12)$$

$$\sin. C=\frac{2}{\sin. a \sin. b}\sqrt{\sin. s \sin.(s-a) \sin.(s-b) \sin.(s-c)} \quad (13)$$

Also, by equation (a), we have

$$1+\cos. A=\frac{2 \sin. s \sin.(s-a)}{\sin. b \sin. c};$$

or

$$2 \cos.^2 \tfrac{1}{2}A=\frac{2 \sin. s \sin.(s-a)}{\sin. b \sin. c}.$$

$$\therefore \cos. \tfrac{1}{2}A=\sqrt{\frac{\sin. s \sin.(s-a)}{\sin. b \sin. c}} \quad . \; . \quad (14)$$

In like manner,

$$\cos. \tfrac{1}{2}B=\sqrt{\frac{\sin. s \sin.(s-b)}{\sin. a \sin. c}} \quad . \; . \quad (15)$$

$$\cos. \tfrac{1}{2}C=\sqrt{\frac{\sin. s \sin.(s-c)}{\sin. a \sin. b}} \quad . \; . \quad (16)$$

Again, by equation (b),

$$1-\cos. A=\frac{2 \sin.(s-c) \sin.(s-b)}{\sin. b \sin. c};$$

or

$$2 \sin.^2 \tfrac{1}{2}A=\frac{2 \sin.(s-b) \sin.(s-c)}{\sin. b \sin. c}.$$

Hence

$$\sin. \tfrac{1}{2}A=\sqrt{\left(\frac{\sin.(s-b) \sin.(s-c)}{\sin. b \sin. c}\right)} \quad . \quad (17)$$

Also

$$\sin. \tfrac{1}{2}B=\sqrt{\left(\frac{\sin.(s-a) \sin.(s-c)}{\sin. a \sin. c}\right)} \quad . \quad (18)$$

$$\sin. \tfrac{1}{2}C=\sqrt{\left(\frac{\sin.(s-a) \sin.(s-b)}{\sin. a \sin. b}\right)} \quad . \quad (19)$$

Again: if we divide equations (17), (18), (19), by equations (14), (15), (16) respectively, we shall obtain,

$$\tan. \tfrac{1}{2}A=\sqrt{\frac{\sin.(s-b) \sin.(s-c)}{\sin. s \sin.(s-a)}} \quad . \quad (20)$$

$$\tan.\tfrac{1}{2}B=\sqrt{\frac{\sin.(s-a)\sin.(s-c)}{\sin.s\sin.(s-b)}} \quad . \quad (21)$$

$$\tan.\tfrac{1}{2}C=\sqrt{\frac{\sin.(s-a)\sin.(s-b)}{\sin.s\sin.(s-c)}} \quad . \quad (22)$$

PROBLEM VIII.

To express the sine of a side of a spherical triangle in terms of the sines and cosines of the angles.

Solution. By (6):

$$\cos.a=\frac{\cos.A+\cos.B\cos.C}{\sin.B\sin.C}.$$

Add unity to both members, and we have

$$1+\cos.a=\frac{\cos.A+\cos.B\cos.C+\sin.B\sin.C}{\sin.B\sin.C};$$

$$=\frac{\cos.A+\cos.(B-C)}{\sin.B\sin.C};$$

$$=\frac{2\cos.\tfrac{1}{2}(A+B-C)\cos.\tfrac{1}{2}(A+C-B)}{\sin.B\sin.C}.$$

Put $s'=\tfrac{1}{2}(A+B+C)$; then $s'-A=\tfrac{1}{2}(B+C-A)$; $s'-B=\tfrac{1}{2}(A+C-B)$; $s'-C=\tfrac{1}{2}(A+B-C)$; therefore,

$$1+\cos.a=\frac{2\cos.(s'-C)\cos.(s'-B)}{\sin.B\sin.C} \quad . \quad . \quad (a)$$

Resuming the expression for cos. a, and subtracting each member from 1, we have

$$1-\cos.a=-\frac{\cos.B\cos.C-\sin.B\sin.C+\cos.A}{\sin.B\sin.C};$$

$$=-\frac{\cos.(B+C)+\cos.A}{\sin.B\sin.C};$$

$$=-\frac{2\cos.\tfrac{1}{2}(A+B+C)\cos.\tfrac{1}{2}(B+C-A)}{\sin.B\sin.C};$$

$$=-\frac{2\cos.s'\cos.(s'-A)}{\sin.B\sin.C} \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Multiply equations (a) and (b), and we shall have

$$1-\cos.^2 a=-\frac{4\cos.s'\cos.(s'-A)\cos.(s'-B)\cos.(s'-C)}{\sin.^2 B\sin.^2 C};$$

therefore,

$$\sin. a = \frac{2}{\sin. B \sin. C} \times \sqrt{-\cos. s' \cos. (s'-A) \cos. (s'-B) \cos. (s'-C)} \qquad (23)$$

In a similar manner,

$$\sin. b = \frac{2}{\sin. A \sin. C} \times \sqrt{-\cos. s' \cos. (s'-A) \cos. (s'-B) \cos. (s'-C)} \qquad (24)$$

$$\sin. c = \frac{2}{\sin. A \sin. B} \times \sqrt{-\cos. s' \cos. (s'-A) \cos. (s'-B) \cos. (s'-C)} \qquad (25)$$

By equation (a) we have

$$1+\cos. a = \frac{2 \cos. (s'-B) \cos. (s'-C)}{\sin. B \sin. C};$$

$$\therefore 2 \cos.^2 \tfrac{1}{2}a = \frac{2 \cos. (s'-B) \cos. (s'-C)}{\sin. B \sin. C};$$

$$\therefore \cos. \tfrac{1}{2}a = \sqrt{\frac{\cos. (s'-B) \cos. (s'-C)}{\sin. B \sin. C}} \quad . \quad . \quad (26)$$

$$\text{Also } \cos. \tfrac{1}{2}b = \sqrt{\frac{\cos. (s'-A) \cos. (s'-C)}{\sin. A \sin. C}} \quad . \quad (27)$$

$$\cos. \tfrac{1}{2}c = \sqrt{\frac{\cos. (s'-A) \cos. (s'-B)}{\sin. A \sin. B}} \quad . \quad (28)$$

By equation (b),

$$1-\cos. a = -\frac{2 \cos. s' \cos. (s'-A)}{\sin. B \sin. C};$$

$$\therefore 2 \sin.^2 \tfrac{1}{2}a = \frac{-2 \cos. s' \cos. (s'-A)}{\sin. B \sin. C};$$

$$\text{hence} \quad \sin. \tfrac{1}{2}a = \sqrt{\frac{-\cos. s' \cos. (s'-A)}{\sin. B \sin. C}} \quad . \quad . \quad (29)$$

$$\text{also} \quad \sin. \tfrac{1}{2}b = \sqrt{\frac{-\cos. s' \cos. (s'-B)}{\sin. A \sin. C}} \quad . \quad . \quad (30)$$

$$\sin. \tfrac{1}{2}c = \sqrt{\frac{-\cos. s' \cos. (s'-C)}{\sin. A \sin. B}} \quad . \quad . \quad (31)$$

Again, by dividing formulas (29, 31) by (26–28), we have

$$\tan. \tfrac{1}{2}a = \sqrt{\frac{-\cos. s' \cos. (s'-A)}{\cos. (s'-B) \cos. (s'-C)}} \quad . \quad . \quad (32)$$

$$\tan.\tfrac{1}{2}b=\sqrt{\frac{-\cos.s'\cos.(s'-B)}{\cos.(s'-A)\cos.(s'-C)}} \quad . \quad (33)$$

$$\tan.\tfrac{1}{2}c=\sqrt{\frac{-\cos.s'\cos.(s'-C)}{\cos.(s'-A)\cos.(s'-B)}} \quad . \quad (34)$$

Although the formulas (23–25), (29–34), appear under an impossible form, they are, in reality, always possible.

For, since the sum of the angles of a spherical triangle is always greater than *two* and less than *six* right angles (Spherical Geom., Prop. 9), therefore

$$A+B+C>180° \text{ and } <540°;$$

therefore $\tfrac{1}{2}(A+B+C)>90°$ and $<270°$;

hence $\cos.s'$ is always *negative*, and $-\cos.s'$ is always *positive.*

Again, if a', b', c' be the three sides of the polar triangle, then, because the sum of any two sides of a spherical triangle is greater than the other side,

$$b'+c'>a';$$

$$\therefore 180°-B+180°-C>180°-A;$$

hence $B+C-A<180°$,

and $\tfrac{1}{2}(B+C-A)<90°$;

consequently, $\cos.(s'-A)$ is always positive. In like manner we can prove that $\cos.(s'-B)$ and $\cos.(s'-C)$ are always positive.

Hence the foregoing formulas are *possible* in every case.

PROPOSITION I.

In every isosceles spherical triangle, the angles opposite the equal sides are equal.

(1) $$\cos.A=\frac{\cos.a-\cos.b\cos.c}{\sin.b\sin.c};$$

$$\cos.B=\frac{\cos.b-\cos.a\cos.c}{\sin.a\sin.c}.$$

But if the triangle is isosceles, $a=b$;

$$\therefore \cos.A=\frac{\cos.a-\cos.a\cos.c}{\sin.a\sin.c};$$

and $$\cos.B=\frac{\cos.a-\cos.a\cos.c}{\sin.a\sin.c};$$

$$\therefore A=B.$$

PROPOSITION II.

Conversely, if the angles of a spherical triangle are equal to each other, the triangle is isosceles.

(6)
$$\cos. a=\frac{\cos. A+\cos. B \cos. C}{\sin. B \sin. C};$$

$$\cos. b=\frac{\cos. B+\cos. A \cos. C}{\sin. A \sin. C}.$$

But
$$B=A;$$

$$\therefore \cos. a=\frac{\cos. A+\cos. A \cos. C}{\sin. A \sin. C};$$

$$\therefore \cos. b=\frac{\cos. A+\cos. A \cos. C}{\sin. A \sin. C};$$

$$\therefore a=b.$$

CHAPTER II.

THEOREM II.

The sines of the angles of a spherical triangle are to each other as the sines of the sides opposite to them.

Demonstration.

$$\text{By (11), (12), } \sin. A=\frac{2\sqrt{\sin. s \sin. (s-a) \sin. (s-b) \sin. s-c)}}{\sin. b \sin. c};$$

$$\sin. B=\frac{2\sqrt{\sin. s \sin. (s-a) \sin. (s-b) \sin. (s-c)}}{\sin. a \sin. c}.$$

Dividing (11) by (12), we have

$$\frac{\sin. A}{\sin. B}=\frac{\sin. a \sin. c}{\sin. b \sin. c}=\frac{\sin. a}{\sin. b} \quad . \quad . \quad (35)$$

In like manner we obtain

$$\frac{\sin. A}{\sin. C}=\frac{\sin. a \sin. b}{\sin. c \sin. b}=\frac{\sin. a}{\sin. c} \quad . \quad . \quad (36)$$

$$\frac{\sin. B}{\sin. C}=\frac{\sin. b \sin. a}{\sin. c \sin. a}=\frac{\sin. b}{\sin. c} \quad . \quad . \quad (37)$$

PROBLEM I.

To express the tangent of the sum and difference of two angles of a spherical triangle in terms of the sides opposite to these angles, and the third angle of the triangle.

Solution. By (1), (3):

$$\cos.\,A=\frac{\cos.\,a-\cos.\,b\,\cos.\,c}{\sin.\,b\,\sin.\,c}\quad\cdot\ \cdot\ \cdot\ \cdot\ \cdot\quad(a)$$

$$\cos.\,C=\frac{\cos.\,c-\cos.\,a\,\cos.\,b}{\sin.\,a\,\sin.\,b};$$

$$\therefore\ \cos.\,c=\cos.\,a\,\cos.\,b+\sin.\,a\,\sin.\,b\,\cos.\,C\quad\cdot\ \cdot\quad(b)$$

Substitute this value of cos. c in equation (1), we have,

$$\cos.\,A=\frac{\cos.\,a-\cos.\,a\,\cos.^2\,b-\cos.\,b\,\sin.\,a\,\sin.\,b\,\cos.\,C}{\sin.\,b\,\sin.\,c};$$

$$=\frac{\cos.\,a\,(1-\cos.^2\,b)-\cos.\,b\,\sin.\,a\,\sin.\,b\,\cos.\,C}{\sin.\,b\,\sin.\,c};$$

$$=\frac{\cos.\,a\,\sin.\,b-\cos.\,b\,\sin.\,a\,\cos.\,C}{\sin.\,c}\quad\cdot\ \cdot\ \cdot\quad(c)$$

Also, by substituting the value of cos. c in equation (b), in the formula (2), we shall have

$$\cos.\,B=\frac{\cos.\,b\,\sin.\,a-\cos.\,a\,\sin.\,b\,\cos.\,C}{\sin.\,c}\quad\cdot\ \cdot\quad(d)$$

Adding equations (c) and (d), we have

$$\cos.\,A+\cos.\,B=$$

$$\frac{\sin.\,a\,\cos.\,b+\sin.\,b\,\cos.\,a-(\sin.\,a\,\cos.\,b+\sin.\,b\,\cos.\,a)\,\cos.\,C}{\sin.\,c};$$

$$\cos.\,A+\cos.\,B=\frac{\sin.\,(a+b)-\sin.\,(a+b)\,\cos.\,C}{\sin.\,c}.\quad\cdot\quad(e)$$

But, by (35), $\dfrac{\sin.\,A}{\sin.\,B}=\dfrac{\sin.\,a}{\sin.\,b}$;

whence, by the theory of proportions,

$$\frac{\sin.\,A\pm\sin.\,B}{\sin.\,B}=\frac{\sin.\,a\pm\sin.\,b}{\sin.\,b}.$$

$$\therefore\ \sin.\,A\pm\sin.\,B=(\sin.\,a\pm\sin.\,b)\,\frac{\sin.\,B}{\sin.\,b};$$

$$=(\sin.\,a\pm\sin.\,b)\,\frac{\sin.\,C}{\sin.\,c}\ .\quad(f)$$

Dividing eq. (f) by (e), and taking first the positive sign,

$$\frac{\sin. A+\sin. B}{\cos. A+\cos. B}=\frac{\sin. a+\sin. b}{\sin. (a+b)}\cdot\frac{\sin. C}{1-\cos. C}.$$

$$\therefore\frac{2\sin. \tfrac{1}{2}(A+B)\cos. \tfrac{1}{2}(A-B)}{2\cos. \tfrac{1}{2}(A+B)\cos. \tfrac{1}{2}(A-B)}=$$

$$\frac{2\sin. \tfrac{1}{2}(a+b)\cos. \tfrac{1}{2}(a-b)}{2\sin. \tfrac{1}{2}(a+b)\cos. \tfrac{1}{2}(a+b)}\cot. \tfrac{1}{2}C.$$

$$\therefore\tan. \tfrac{1}{2}(A+B)=\frac{\cos. \tfrac{1}{2}(a-b)}{\cos. \tfrac{1}{2}(a+b)}\cot. \tfrac{1}{2}C \quad . \quad . \quad (38)$$

Again, by taking the negative sign, we shall have

$$\frac{\sin. A-\sin. B}{\cos. A+\cos. B}=\frac{\sin. a-\sin. b}{\sin. (a+b)}\cdot\frac{\sin. C}{1-\cos. C};$$

or

$$\frac{2\sin. \tfrac{1}{2}(A-B)\cos. \tfrac{1}{2}(A+B)}{2\cos. \tfrac{1}{2}(A-B)\cos. \tfrac{1}{2}(A+B)}=$$

$$\frac{2\sin. \tfrac{1}{2}(a-b)\cos. \tfrac{1}{2}(a+b)}{2\sin. \tfrac{1}{2}(a+b)\cos. \tfrac{1}{2}(a+b)}\cdot\frac{\sin. C}{1-\cos. C};$$

therefore $\tan. \tfrac{1}{2}(A-B)=\dfrac{\sin. \tfrac{1}{2}(a-b)}{\sin. \tfrac{1}{2}(a+b)}\cot. \tfrac{1}{2}C$. (39)

Formulas (38) and (39) may be written thus:

$$\cos. \tfrac{1}{2}(a+b)\tan. \tfrac{1}{2}(A+B)=\cos. \tfrac{1}{2}(a-b)\cot. \tfrac{1}{2}C;$$

$$\sin. \tfrac{1}{2}(a+b)\tan. \tfrac{1}{2}(A-B)=\sin. \tfrac{1}{2}(a-b)\cot. \tfrac{1}{2}C.$$

That is, *in any spherical triangle, as the cosine of half the sum of two sides is to the cosine of half their difference, so is the cotangent of half the included angle to the tangent of half the sum of the other two angles.* And

As the sine of half the sum of two sides is to the sine of half their difference, so is the cotangent of half the included angle to the tangent of half the difference of the other two angles.

This is the first of *Napier's Analogies.*

In a manner precisely similar will be found

$$\tan. \tfrac{1}{2}(B+C)=\frac{\cos. \tfrac{1}{2}(b-c)}{\cos. \tfrac{1}{2}(b+c)}\cot. \tfrac{1}{2}A \quad . \quad (40)$$

$$\tan. \tfrac{1}{2}(B-C)=\frac{\sin. \tfrac{1}{2}(b-c)}{\sin. \tfrac{1}{2}(b+c)}\cot. \tfrac{1}{2}A \quad . \quad (41)$$

$$\tan. \tfrac{1}{2}(A+C)=\frac{\cos. \tfrac{1}{2}(a-c)}{\cos. \tfrac{1}{2}(a+c)}\cot. \tfrac{1}{2}B \quad . \quad (42)$$

$$\tan. \tfrac{1}{2}(A-C)=\frac{\sin. \tfrac{1}{2}(a-c)}{\sin. \tfrac{1}{2}(a+c)}\cot. \tfrac{1}{2}B \quad . \quad (43)$$

PROBLEM II.

To express the tangent of the sum and difference of two sides of a spherical triangle in terms of the angles opposite to them and the third side of the triangle.

Solution. Let A, B, C, a, b, c, be the angles and sides of a spherical triangle; A′, B′, C′, a', b', c', the corresponding parts of the polar triangle; then, by (38),

$$\tan.\tfrac{1}{2}(A'+B')=\frac{\cos.\frac{1}{2}(a'-b')}{\cos.\frac{1}{2}(a'+b')}\cot.\tfrac{1}{2}C'.$$

$$\therefore \tan.\tfrac{1}{2}\{(180^\circ-a)+(180^\circ-b)\}=$$

$$\frac{\cos.\frac{1}{2}\{(180^\circ-A)-(180^\circ-B)\}}{\cos.\frac{1}{2}\{(180^\circ-A)+(180^\circ-B)\}}\cot.\tfrac{1}{2}(180^\circ-c);$$

or

$$\therefore \tan.\tfrac{1}{2}(a+b)=\frac{\cos.\frac{1}{2}(A-B)}{\cos.\frac{1}{2}(A+B)}\tan.\tfrac{1}{2}c \quad . \quad . \quad . \quad (44)$$

Again, by (39):

$$\tan.\tfrac{1}{2}(A'-B')=\frac{\sin.\frac{1}{2}(a'-b')}{\sin.\frac{1}{2}(a'+b')}\cot.\tfrac{1}{2}C';$$

or

$$\tan.\tfrac{1}{2}\{(180^\circ-a)-(180^\circ-b)\}=$$

$$\frac{\sin.\frac{1}{2}\{(180^\circ-A)-(180^\circ-B)\}}{\sin.\frac{1}{2}\{(180^\circ-A)+(180^\circ-B)\}}\cot.\tfrac{1}{2}(180^\circ-c).$$

$$\therefore \tan.\tfrac{1}{2}(a-b)=\frac{\sin.\frac{1}{2}(A-B)}{\sin.\frac{1}{2}(A+B)}\tan.\tfrac{1}{2}c \quad . \quad . \quad . \quad (45)$$

In like manner we shall obtain

$$\tan.\tfrac{1}{2}(b+c)=\frac{\cos.\frac{1}{2}(B-C)}{\cos.\frac{1}{2}(B+C)}\tan.\tfrac{1}{2}a \quad . \quad (46)$$

$$\tan.\tfrac{1}{2}(b-c)=\frac{\sin.\frac{1}{2}(B-C)}{\sin.\frac{1}{2}(B+C)}\tan.\tfrac{1}{2}a \quad . \quad (47)$$

$$\tan.\tfrac{1}{2}(a+c)=\frac{\cos.\frac{1}{2}(A-C)}{\cos.\frac{1}{2}(A+C)}\tan.\tfrac{1}{2}b \quad . \quad (48)$$

$$\tan.\tfrac{1}{2}(a-c)=\frac{\sin.\frac{1}{2}(A-C)}{\sin.\frac{1}{2}(A+C)}\tan.\tfrac{1}{2}b \quad . \quad (49)$$

Or, *as the cosine of half the sum of two angles is to the cosine of half their difference, so is the tangent of half the included side to the tangent of half the sum of the other two sides;* and

As the sine of half the sum of two angles is to the sine of half their difference, so is the tangent of half the included side to the tangent of half the difference of those sides.

This is the second of *Napier's Analogies.*

PROBLEM III.

To express the cotangent of an angle of a spherical triangle in terms of the side opposite, one of the other sides, and the angle contained between these two sides.

Solution. From (36), $\sin. c = \frac{\sin. C}{\sin. A} \sin. a$;

and from (eq. c, Prob. 1),

$$\cos. A \sin. c = \cos. a \sin. b - \sin. a \cos. b \cos. C;$$

hence $\cos. A \frac{\sin. C}{\sin. A} \sin. a = \cos. a \sin. b - \sin. a \cos. b \cos. C$;

$$\therefore \cot. A = \cot. a \sin. b \operatorname{cosec.} C - \cos. b \cot. C \quad . \quad . \quad (50)$$

If we substitute the value of cos. b derived from equation (2) in equation (1), we shall find a value for cot. A in terms of a, c, B.

$$\text{Thus } \cot. A = \cot. a \sin. c \operatorname{cosec.} B - \cos. c \cot. B \quad (51)$$

Proceeding in like manner, we shall obtain for the other angles,

$$\cot. B = \cot. b \sin. a \operatorname{cosec.} C - \cos. a \cot. C \quad (52)$$
$$\cot. B = \cot. b \sin. c \operatorname{cosec.} A - \cos. c \cot. A \quad (53)$$
$$\cot. C = \cot. c \sin. a \operatorname{cosec.} B - \cos. a \cot. B \quad (54)$$
$$\cot. C = \cot. c \sin. b \operatorname{cosec.} A - \cos. b \cot. A \quad (55)$$

PROBLEM IV.

To express the cotangent of a side of a spherical triangle in terms of the opposite angle, one of the other angles, and the side interjacent to these two angles.

Solution. Using the same notation as before for the angles and sides of the spherical and polar triangles, we have, by (50),

$$\cot. A' = \cot. a' \sin. b' \operatorname{cosec.} C' - \cos. b' \cot. C';$$
$$\therefore \cot. (180^\circ - a) = \cot. (180^\circ - A) \sin. (180^\circ - B) \operatorname{cosec.} (180^\circ - c) - \cos. (180^\circ - B) \cot. (180^\circ - c);$$

or $-\cot. a = -\cot. A \sin. B \operatorname{cosec.} c - \cos. B \cot. c$;

$$\therefore \cot. a = \cot. A \sin. B \operatorname{cosec.} c + \cos. B \cot. c \quad (56)$$

Applying the same process to the other formulas in Prob. III., we shall obtain analogous results. Thus

$$\cot. a = \cot. A \sin. C \operatorname{cosec.} b + \cos. C \cot. b \quad (57)$$
$$\cot. b = \cot. B \sin. A \operatorname{cosec.} c + \cos. A \cot. c \quad (58)$$
$$= \cot. B \sin. C \operatorname{cosec.} a + \cos. C \cot. a \quad (59)$$
$$\cot. c = \cot. C \sin. A \operatorname{cosec.} b + \cos. A \cot. c \quad (60)$$
$$= \cot. C \sin. B \operatorname{cosec.} a + \cos. B \cot. a \quad (61)$$

RIGHT-ANGLED SPHERICAL TRIANGLES.

CHAPTER III.

SPHERICAL TRIANGLES OF ONE RIGHT ANGLE ONLY.

A SPHERICAL triangle consists of six parts; any three of these being given, the rest may be found. In the present case, one of the angles is supposed to be a right angle. If any other two parts be given, the other three may be determined.

Now the combination of five quantities, taken three and three, is equal to

$$\frac{5.4.3}{1.2.3}=10.$$

Therefore *ten* different cases present themselves in the solution of right-angled spherical triangles.

Two rules have been invented, by the help of which we are enabled to solve every case of right-angled spherical triangles. They are known by the name of *Napier's Rules for Circular Parts*, and are as follows:

The right angle is altogether thrown out of consideration. The two sides, the complements of the two angles, and the complement of the hypothenuse, are called *the circular parts.* Any one of these circular parts may be called a *middle part*, and then the two circular parts immediately adjacent to the right and left of the middle part are called *adjacent parts;* the other two remaining circular parts, each separated from the middle part by an adjacent part, are called *opposite parts.* This being stated, we may now give the

RULES.

1. *M being the middle part, the product of sin. M and tabular radius is equal to the product of the tangents of the adjacent parts.*

2. *The product of sin. M and tabular radius is equal to the product of the cosines of the opposite parts.*

In order to make these rules clearly understood, we will show the manner in which they are applied in various cases. It will assist the student to remember these rules by observing that the first syllable of *tangent* and *adjacent* each has the letter a in it, and *cosines* and *opposite* the letter o.

Let A, B, C be a spherical triangle right-angled at C; let a be assumed as the middle part; then

$(90^\circ-B)$ and b are the adjacent parts,

and $(90^\circ-c)$, $(90^\circ-A)$, are the opposite parts.

Then, by rule (1),

$$R\times\sin. a=\tan. (90^\circ-B) \tan. b;$$
$$=\cot. B \tan. b \quad . \quad . \quad . \quad . \quad . \quad (a)$$

By rule (2),

$$R\times\sin. a=\cos. (90^\circ-A) \cos. (90^\circ-c);$$
$$=\sin. a \sin. c \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

2d. Let b be the middle part; then

$(90^\circ-A)$ and a are the adjacent parts,

and $(90^\circ-c)$ and $(90^\circ-B)$ are the opposite parts.

Then, by rule (1),

$$R \sin. b=\tan. (90^\circ-A) \tan. a;$$
$$=\cot. A \tan. a \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Rule (2),

$$R.\sin. b=\cos. (90^\circ-B) \cos. (90^\circ-c);$$
$$=\sin. B \sin. c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

3d. Let $(90^\circ-c)$ be the middle part; then

$(90^\circ-A)$ and $(90^\circ-B)$ are the adjacent parts,

and b and a are the opposite parts.

Then, by rule (1),

$$R.\sin. (90^\circ-c)=\tan. (90^\circ-A) \tan. (90^\circ-B);$$
$$R \cos. c=\cot. A \cot. B \quad . \quad . \quad . \quad . \quad . \quad (e)$$

Rule (2), $$R.\sin. (90^\circ-c)=\cos. a \cos. b;$$
$$R \cos. c=\cos. a \cos. b \quad . \quad . \quad (f)$$

4th. Let $(90^\circ-A)$ be the middle part; then
$(90^\circ-c)$ and b are the adjacent parts,
and $(90^\circ-B)$ and a are the opposite parts.

Then, by rule (1),

$$R \sin.(90^\circ-A)=\tan.(90^\circ-c)\tan. b;$$

$$R \cos. A=\cot. c \tan. b \;.\;.\;.\;.\;.\quad (g)$$

Rule (2), $$R \sin.(90^\circ-A)=\cos.(90^\circ-B)\cos. a;$$

$$R \cos. A=\sin. B \cos. a \;.\;.\quad (h)$$

5th. Let $(90^\circ-B)$ be the middle part; then
$(90^\circ-c)$ and a are the adjacent parts,
and $(90^\circ-A)$ and b are the opposite parts.

Then, by rule (1),

$$R.\sin.(90^\circ-B)=\tan.(90^\circ-c)\tan. a;$$

$$R \cos. B=\tan. a \cot. c \;.\;.\;.\;.\;.\quad (i)$$

Rule (2), $$R.\sin.(90^\circ-B)=\cos.(90^\circ-A)\cos. b;$$

$$\therefore R \cos. B=\sin. A \cos. b \;.\;.\quad (k)$$

Collecting equations (a to k), we shall have

$R \sin. a = \cot. B \tan. b$ . . . . . . (60)
$R \sin. a = \sin. A \sin. c$ . . . . . . (61)
$R \sin. b = \cot. A \tan. a$ . . . . . . (62)
$R \sin. b = \sin. B \sin. c$ . . . . . . (63)
$R \cos. c = \cot. A \cot. B$ . . . . . . (64)
$R \cos. c = \cos. a \cos. b$ . . . . . . (65)
$R \cos. A = \tan. b \cot. c$ . . . . . . (66)
$R \cos. A = \sin. B \cos. a$ . . . . . . (67)
$R \cos. B = \tan. a \cot. c$ . . . . . . (68)
$R \cos. B = \sin. A \cos. b$ . . . . . . (69)

It now remains to prove that these formulas are true.

Formula (3): $$\cos. C=\frac{\cos. c-\cos. a \cos. b}{\sin. a \sin. b}.$$

But when $C=90^\circ$, $\cos. C=0$;

therefore $$0=\frac{\cos. c-\cos. a \cos. b}{\sin. a \sin. b};$$

or $\cos. c=\cos. a \cos. b$, or $R \cos. c=\cos. a \cos. b$,[*] which is formula (65).

Again, by (36), $$\frac{\sin. a}{\sin. c}=\frac{\sin. A}{\sin. C};$$

but when $C=90^\circ$, $\sin. C=1$; therefore

---

[*] We must multiply by R to make the equation homogeneous.

$$\sin. a = \sin. A \sin. c,$$

which is formula (61).

Also (37), $\frac{\sin. b}{\sin. c} = \frac{\sin. B}{\sin. C}$. But if $C = 90°$, $\sin. C = 1$;

$$\therefore \sin. b = \sin. B \sin. c,$$

which is formula (63).

Also (1), $\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}$.

If we substitute for cos. c its value in (65), we shall have

$$\cos. A = \frac{\cos. a - \cos. a \cos.^2 b}{\sin. b \sin. c} = \frac{\cos. a(1 - \cos.^2 b)}{\sin. b \sin. c} = \frac{\cos. a \sin.^2 b}{\sin. b \sin. c}$$

$$= \frac{\cos. a \sin. b}{\sin. c}.$$

Again, if we substitute for sin. c its value as found in (61), we have

$$\cos. A = \frac{\cos. a \sin. b}{\frac{\sin. a}{\sin. A}} = \frac{\cos. a \sin. b \sin. A}{\sin. a};$$

$$\therefore \sin. b = \cot. A \tan. a,$$

which is formula (62).

Again, $\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}$.

Substituting for cos. a its value in (65), we have

$$\cos. A = \frac{\frac{\cos. c}{\cos. b} - \cos. b \cos. c}{\sin. b \sin. c}$$

$$= \frac{\cos. c \sin. b}{\sin. c \cos. b}$$

$$= \tan. b \cot. c,$$

which is formula (66).

Again (2), $\cos. B = \frac{\cos. b - \cos. a \cos. c}{\sin. a \sin. c}$.

Substituting for cos. c its value in (65), then

$$\cos. B = \frac{\cos. b - \cos. b \cos.^2 a}{\sin. a \sin. c}$$

$$= \frac{\cos. b \sin. a}{\sin. c}.$$

Now, substituting for sin. c its value in (63), we have

$$\text{cos. B}=\frac{\dfrac{\text{cos. }b\text{ sin. }a}{\text{sin. }b}}{\text{sin. B}};$$

$$\therefore\ \text{sin. }a=\text{cot. B tan. }b,$$

which is formula (60).

Again, $$\text{cos. B}=\frac{\text{cos. }b-\text{cos. }a\text{ cos. }c}{\text{sin. }a\text{ sin. }c}.$$

Substituting for cos. b its value in (65), we shall have

$$\text{cos. B}=\frac{\dfrac{\text{cos. }c}{\text{cos. }a}-\text{cos. }a\text{ cos. }c}{\text{sin. }a\text{ sin. }c}$$

$$=\frac{\text{cos. }c\text{ sin. }a}{\text{sin. }c\text{ cos. }a};$$

$$\therefore\ \text{cos. B}=\text{tan. }a\text{ cot. }c,$$

which is formula (68).

Again, by (6), $$\text{cos. }a=\frac{\text{cos. A}+\text{cos. B cos. C}}{\text{sin. B sin. C}}.$$

But when C=90°, cos. C=0, and sin. C=1;

$$\therefore\ \text{cos. }a=\frac{\text{cos. A}}{\text{sin. B}};$$

hence $$\text{cos. A}=\text{sin. B cos. }a,$$

which is formula (67).

And, by (7), $$\text{cos. }b=\frac{\text{cos. B}+\text{cos. A cos. C}}{\text{sin. A sin. C}}.$$

But when C=90°,

$$\text{cos. }b=\frac{\text{cos. B}}{\text{sin. A}};$$

$$\therefore\ \text{cos. B}=\text{sin. A cos. }b,$$

which is formula (69).

Lastly, by (8), $$\text{cos. }c=\frac{\text{cos. C}+\text{cos. A cos. B}}{\text{sin. A sin. B}}.$$

When C=90°,

$$\text{cos. }c=\frac{\text{cos. A cos. B}}{\text{sin. A sin. B}};$$

$$\therefore\ \text{cos. }c=\text{cot. A cot. B},$$

which is formula (64).

We have thus proved the results derived from Napier's Rules to be true, and may therefore, without hesitation, apply those rules to the solution of the various problems in right-angled spherical triangles.

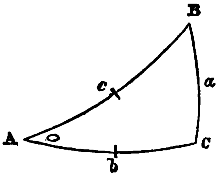
SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

In the solution of right-angled spherical triangles, we take the *two given parts* to find the other three parts in succession.

As the student sometimes finds a difficulty in ascertaining which is the *middle part* when he is applying Napier's Circular Parts, we shall give a few examples to illustrate the subject more fully.

1°. *When we have the hypothenuse, c, and the base, b, given, to find the other parts.*

To find the angle A.



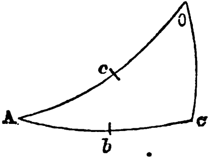
Here the three parts lie together; therefore the complement of the angle A is the middle part, and the extremes *are adjacent.*

$\therefore$ R.cos. A$=$tan. b cot. c.

Now, as the middle part A is the unknown quantity, we begin the proportion with radius.

As Radius : tan. b :: cot. c : cos. A;
whence the angle A is determined.

To find the angle B.



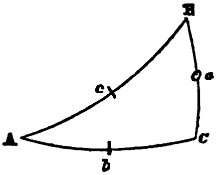
Here two of the parts, c and B, are adjacent to each other, but the side b is separated from both; hence b is the middle part, and the extremes are *opposite.*

$\therefore$ R.sin. $b$$=$sin. c sin. B;

Or, putting the equation into a proportion,

As $\quad \sin. c : \text{Radius} :: \sin. b : \sin. B$;
whence the angle B is determined.

To find the side a.



Here the two parts a and b are adjacent, because the right angle is altogether thrown out of consideration; but the hypothenuse c is separated from both; therefore the complement of c is the middle part, and the extremes are *opposite*.

$$R.\cos. c = \cos. a.\cos. b;$$

or, as $\quad \cos. b : \text{Radius} :: \cos. c : \cos. a$.

Hence all the parts are determined from the two parts that were given.

When a required part is determined from its sine, there will be two angles, both of which may be regarded as solutions, except when the ambiguity is removed by either of the following principles.

(1.) *In a right-angled spherical triangle, an angle and its opposite are always in the same quadrant; that is, they are either both less or both greater than* 90°.

(2.) *When the two sides, including the right angle, are in the same quadrant, the hypothenuse is less than* 90°; *and when the two sides are in different quadrants the hypothenuse is greater than* 90°.

EXAMPLES.

1. In the right-angled triangle ABC are given the hypothenuse $c = 140^\circ$, and the perpendicular $a = 20^\circ$, to find the other parts. *Ans.* $A = 32^\circ\ 8'\ 48''$.
$B = 115^\circ\ 42'\ 23''$.
$b = 144^\circ\ 36'\ 28''$.

2. Given $A = 80^\circ\ 10'\ 30''$, $b = 155^\circ\ 46'\ 42''$, to find the other parts. *Ans.* $a = 67^\circ\ 6'\ 52''$.
$b = 110^\circ\ 46'\ 20''$.
$B = 155^\circ\ 58'\ 24''$.

3. Given $a = 116^\circ$, $b = 16^\circ$ to find the other parts.
Ans. $A = 97^\circ\ 39'\ 24''$.
$B = 17^\circ\ 41'\ 40''$.
$c = 114^\circ\ 55'\ 20''$.

4. Given A=60° 47′ 24″, B=57° 16′ 20″, to find the other parts.

Ans. $a=54° 32' 33''$.
$b=51° 43' 36''$.
$c=68° 56' 29''$.

SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

The Problems and Theorems in the preceding pages on Spherical Trigonometry are sufficient to solve every triangle, or to obtain any part whatever of the triangle we may wish to obtain without going through the calculations of all the other parts. We shall only give the six different cases under which the spherical triangle may present itself.

1°. *When two sides and an angle opposite one of them are given.*

From Chap. II., th. 2, we derive the following

RULE.

As *the sine of the side opposite the given angle* : *the sine of the side opposite the required angle* :: *the sine of the given angle* : *the sine of the required angle.*

We may then find the other angle from Prob. I., formula 39, and the other side from Prob. II., formula 45.

Or we may let fall a perpendicular from the unknown angle on the unknown side, and determine them by Napier's rules for circular parts.

By the above rule we shall find two values for the angle supplementary to each other. If these two values, when used in formulas (39) and (45), cause neither of them to become negative, there will be two solutions.

2°. *When two angles and a side opposite one of them are given.*

Let A, B, and a be given.

First find b by the equation

$$\frac{\sin. A}{\sin. B}=\frac{\sin. a}{\sin. b},$$

as before. Thus,

As *the sine of the angle opposite the given side* : *the sine of the angle opposite the required side* :: *the sine of the given side* : *the sine of the required side.*

We may then find C and c by using the same formulas as in the last case.

We shall, as before, find two values for the side b supplementary to each other.

If these two values, when substituted in formulas (39) and (45), cause neither to become negative, there will be two solutions.

We may, as in the last case, let fall a perpendicular from the unknown angle on the unknown side, and find C and c by *Napier's Circular Parts.*

That we may distinguish the trigonometric functions of parts greater than 90° from those less than 90°, we should carefully attend to the algebraic signs, using the *plus* and *minus* to the right of the logarithms.

3°. *When two sides and the included angle are given.*

Use the first of *Napier's Analogies*, p. 209. Then, *to half the sum of the two angles add half their difference for the greater angle; and from half the sum of the two angles subtract half their difference for the less angle.*

The third side can then be found by (1).

4°. *When two angles and the interjacent side are given.*

Use the second of Napier's Analogies, Prob. II., p. 210. Then find each side separately by addition and subtraction, as in (3).

5°. *When the three sides are given.*

Any of the formulas from (14) to (22) will give the angles.

If we use $\tan.\ \frac{1}{2}\mathrm{A}=\sqrt{\dfrac{\sin.\ (s-b)\sin.\ (s-c)}{\sin.\ s.\sin.\ (s-a)}}$,

where $s=\frac{1}{2}(a+b+c)$, we shall have

$$\log.\tan.\ \tfrac{1}{2}\mathrm{A}=\tfrac{1}{2}\{\log.\sin.\ (s-b)+\log.\sin.\ (s-c)\\ -\log.\sin.\ s-\log.\sin.\ (s-a).$$

Now, since these formulas give the values of the half angles, there will be no ambiguity in this case.

6°. *When the three angles are given.*

Any of the formulas from (29) to (34) will give the sides.

Using formula (26), we have

$$2 \log. \cos. \tfrac{1}{2}a = \log. \cos. (s' - B) + \log. \cos. (s' - C)$$
$$- \log. \sin. B - \log. \sin. C,$$

where $s' = \frac{1}{2}(A + B + C)$.

As in (5), there can be no ambiguity in the values of a, b, and c.

EXAMPLES.

1. Given the angle $B = 95° \, 30'$, the side $c = 64° \, 30'$, the side $b = 100°$, to find the other parts.

Ans. $A = 98° \, 32' \, 24''$.
$C = 65° \, 49' \, 25''$.
$a = 101° \, 55' \, 58''$.

2. Given the angle $A = 61° \, 37' \, 52''$, $B = 139° \, 54' \, 34''$, and the side $b = 150° \, 17' \, 26''$, to find the other parts.

Ans. $a = 114° \, 26' \, 49''$.
$c = 82° \, 33' \, 31''$.
$C = 79° \, 10' \, 30''$.

3. Given the side $b = 99° \, 40' \, 48''$, the side $c = 100° \, 49' \, 30''$, and the angle $A = 65° \, 33' \, 10''$, to find the other parts.

Ans. $a = 64° \, 23' \, 15''$.
$B = 95° \, 38' \, 4''$.
$C = 97° \, 26' \, 29''$.

4. Given the angle $A = 95° \, 38' \, 4''$, the angle $C = 97° \, 26' \, 29''$, and the interjacent side $b = 64° \, 23' \, 15''$, to find the other parts.

Ans. $a = 99° \, 40' \, 48''$.
$c = 100° \, 49' \, 30''$.
$B = 65° \, 33' \, 10''$.

5. Given the side $a = 100°$, the side $b = 50°$, and the side $c = 60°$, to find the angles.

Ans. $A = 138° \, 15' \, 45''$.
$B = 31° \, 11' \, 14''$.
$C = 35° \, 49' \, 58''$.

6. Given the angle $A = 115° \, 36' \, 45''$, the angle $B = 80° \, 19' \, 12''$, and the angle $C = 79° \, 10' \, 30''$, to find the sides.

Ans. $a = 114° \, 26' \, 50''$.
$b = 84° \, 21' \, 56''$.
$c = 82° \, 33' \, 30''$.

SURVEYING.

(1.) The object of surveying, as we now propose to treat of it, is to measure, lay out, and divide portions of land, the boundaries of which are so small that they may be considered as lines on a *plane surface.* To these operations Plane Trigonometry can be applied; but when more extensive portions of the earth's surface are to be measured, recourse must be had to the principles of Spherical Trigonometry. The method then is called *Geodesy.*

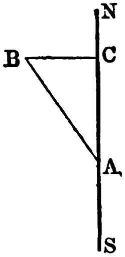
DEFINITIONS.

1. The *distance* of a line in surveying is its length measured in a *horizontal* direction.

2. A *meridian* is any line running due north and south.

3. The *bearing* of a line is the angle which it makes with a meridian.

4. The *difference of latitude* of a line is the distance that one end is farther north or south than the other end.



Thus, if NS be a meridian passing through the end A of the line AB, and BC perpendicular to NS, then AC is the difference of latitude of AB.

When the difference of latitude is north, it is called *plus;* when south, *minus.*

5. The *departure* of a line is the distance that one end is farther east or west than the other end.

In the above diagram, CB is the departure of the line AB.

The distance, AB, the departure, BC, and the difference of latitude, CA, evidently form a right-angled triangle.

6. The *meridian distance of any station* is its perpendicular distance from a meridian passing through any assumed point.

7. The *meridian distance of a line* is the distance of the *middle point* of that line from the assumed meridian.

8. The *Traverse Table* is a table containing the differences of latitude and departures, computed for the bearings from $\frac{1}{4}$ degree to 90 degrees, and distances from 1 to 100, by the formulas (Trig., Prop. 1)

$$\text{dif. of lat.} = \text{distance} \times \text{cos. bearing.}$$
$$\text{departure} = \text{distance} \times \text{sin. bearing.}$$

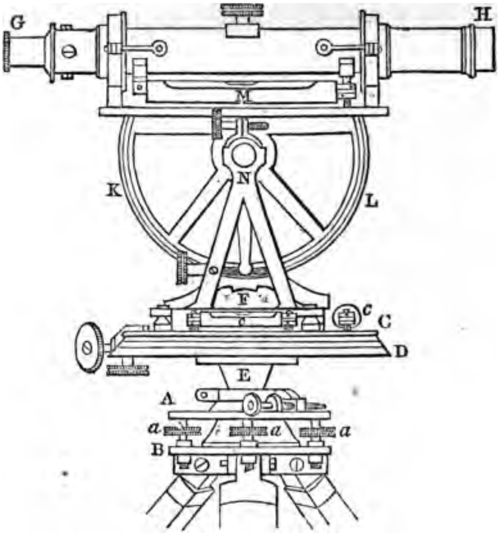
Or, in *logs.*,

$$\text{log. dif. of lat.} = \text{log. distance} + \text{log. cos. bearing} - 10.$$
$$\text{log. departure} = \text{log. distance} + \text{log. sin. bearing} - 10.$$

9. The *area* or *contents* of a tract of land is the ***horizontal surface*** included within its boundaries, expressed in known terms, as *Acres*, *Roods*, and *Perches.*

MEASUREMENT OF ANGLES.

(2.) The instruments generally used for measuring angles are the Surveyor's Compass, Theodolite, and Transit. We shall describe only *The Theodolite.*



The Theodolite is an instrument used to measure horizontal and vertical angles. It has two circular brass plates, C and D, the former of which is called the *vernier* plate, and the latter the *graduated limb*. The vernier plate turns freely upon the lower, and both have a horizontal motion by means of the vertical axis, E. This axis consists of two parts, external and internal; the former secured to the graduated limb, D, and the latter to the vernier plate, C. Their form is conical, nicely fitted and ground into each other, having an easy and steady motion. The external centre also fits into a ball, and the parts are held together by a screw at the lower end of the internal axis.

The diameter of the lower plate is greater than that of the upper one, and its edge is chamfered and silvered to receive the graduations. On opposite parts of the edge of the upper plate a short space is also chamfered, forming with the edge of the lower plate a continued inclined plane. These spaces are also silvered, and form the *verniers*. The lower plate is usually graduated to thirty minutes of a degree, and it is subdivided by the vernier to single minutes, which, being read off through the microscope, half or quarter minutes can easily be estimated.

The parallel plates A and B are held together by a ball and socket, and are set firm and parallel to each other by four milled screws, three of which, *a a a*, are shown in the diagram; these turn in sockets fixed to the lower plate, while their heads press against the under side of the upper plate, and, being set in pairs opposite each other, they act in contrary directions, by which means the theodolite is leveled for observation. The whole rests upon a tripod, which, being opened, makes a firm stand. The graduated limb can be fixed in any position by tightening the clamping-screw, which causes the collar to embrace the axis E, and prevents its moving; but, it being requisite that it should be fixed in some precise position more exactly than can be done by the hand alone, the whole instrument, when thus clamped, can be moved any small quantity by means of the screw which is attached to the upper parallel plate. In like manner, the vernier plate can be fixed to the graduated limb in any position by a clamp which is also furnished with a slow motion, the screw of which is generally called the tangent screw. Upon the vernier plate two spirit levels, *c c*, are placed at right angles to each other, with their proper adjusting screws: their use is to determine when the graduated limb is horizontal or perfectly level. A compass is also placed at F.

The frames, one of which is seen at N, support the pivots of the horizontal axis of the vertical semicircle LK, on which the telescope is placed. The arm which bears the microscope for reading the altitudes or depressions measured by the semicircle, and denoted by the vernier, has a motion between the bars of the frame N, and can be moved before the vernier for reading of the minutes. Another screw clamps the opposite end of the horizontal axis, and, by a tangent screw of slow motion, by which the vertical arc and telescope are moved very small quantities up or down, the contact can be made perfect when an observation is taken. One side of the vertical semicircle is divided into degrees and half degrees, which are farther divided into single minutes by the aid of its vernier. The graduation commences with *zero* at the middle of the semicircle, and reads both ways to 90°. Under and parallel to the telescope is a spirit level, M, to show when the telescope is brought to a horizontal position. In the focus of the telescope two lines, called wires, are fixed at right angles to each other, to enable us with precision to direct the telescope upon any object we desire.

HEIGHTS AND DISTANCES.

(3.) By the mensuration and protraction of lines and angles, the lengths, heights, depths, and distances of objects are determined.

Accessible lines are measured by applying to them some certain measure a number of times; but inaccessible lines must be measured by taking angles on some method drawn from the principles of Geometry or Trigonometry.

When instruments are used for taking the magnitude of the angles in degrees, the lines are then calculated by

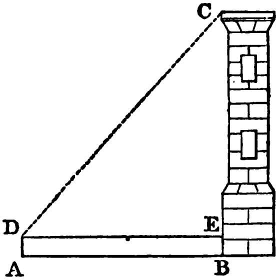
Trigonometry; in the other methods, the lines are calculated from the principles of similar triangles.

Angles of elevation or depression are usually taken with the theodolite or quadrant. When thus taken, we must add the height of the instrument to obtain the entire height of the object above the plane on which we are standing.

The following examples are to be constructed and calculated by the methods treated of in Plane Trigonometry.

EXAMPLE I.

Having measured a distance of 200 feet in a direct horizontal line from the bottom of a tower, the angle of elevation of its top, taken at that point, was found to be 47° 30′, the instrument being five feet above the ground. Required the height of the tower.



Construction. On the indefinite line, lay off AB=200 from the scale of equal parts. Erect perpendiculars at A and B, and make AD and BE each=5 from the same scale. Join DE, and at D draw DC, making the angle CDE=47° 30′, and the point C, where this line meets the perpendicular BE produced, will be the top of the tower. Then BC will be the height required.

Calculation. As radius . . . 10.000000
: tan. D . . . 10.037947
:: DE 2.301030
CE=218.26 =2.338977;

to which must be added the height of the instrument (five feet), to obtain the height of the tower;

$\therefore$ BC=223.26 feet.

Ex. 2. What was the perpendicular height of a balloon when its angles of elevation were 36° and 64°, as taken by two observers at the same time, both on the same side of it, in the same vertical plane, their distance asunder being 880 yards? *Ans.* 990.268 yds.

Ex. 3. At 170 feet distance from the bottom of a tower, the angle of its elevation was found to be 52° 30′. required the altitude of the tower? *Ans.* 221.5 feet.

Ex. 4. From a window near the bottom of a house, which was on a level with the base of a steeple, I took the angle of elevation of the top of the steeple equal 40°; then from another window, 18 feet directly above the former, the like angle was 37° 30′; required the height of the steeple and its distance from the place of observation.

Ans. { Height, 210.44.
Distance, 250.79.

PROBLEM I.

(4.) *To find the area of an irregular tract of land bounded by straight lines.*

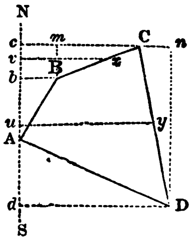
If there are fences, or any other obstructions on the lines, the surveyor should provide himself with a straight pole eight or ten feet in length, with which to measure equal distances at each end of the line. Beginning at any corner of the field, set the pole up perpendicularly at the far end, and the compass or theodolite at the starting-point, each equally distant from the boundary-line. Turn the instrument so that the pole can be seen directly behind the wires or spider-lines of the instrument; then, when the needle settles, the degrees and parts of a degree must be carefully observed, reckoning from that end of the needle which makes the smaller angle with the sight.

This will be the bearing of the first course.

In measuring the distance, great care must be taken by the *chain-bearers* to keep in a straight line, and, when ascending or descending a hill, *to carry the chain horizontally.*

The survey having been actually taken, the area may be computed by means of the difference of latitude and departure corresponding to the bearings and distances of the several sides.

Thus, in the field ABCD, beginning at the most westerly point A, with the bearing and distance of AB, we may ulate the difference of latitude Ab, departure bB, by the formulas



$$\text{dif. of lat.} = \text{dist.} \times \cos. \text{bearing} \quad . \quad . \quad (1)$$
$$\text{departure} = \text{dist.} \times \sin. \text{bearing} \quad . \quad . \quad . \quad (2)$$

Proceed in like manner with each of the other sides, and thus obtain successively bc, mC, nD, Cn, etc.

Instead of calculating each latitude and departure by formulas (1) and (2), it is usual to take them from a table of latitudes and departures prepared for that purpose, which is called the *Traverse Table.*

(5.) In going round a tract of land and returning to the place of beginning, it is evident that the whole northing which has been made must be equal to the whole southing, and the whole easting to the whole westing; or, in other words, that the sum of all the northings must be equal to that of the southings, and the sum of the eastings to that of the westings.

This principle enables us to judge of the accuracy of a survey when the bearings and distances of all the sides have been taken. It is, therefore, *the first place of verification.*

If the sum of the computed northings is equal to the sum of the computed southings, and the sum of the eastings equal to that of the westings, the survey has been *accurately* taken. This, however, is rarely, if ever, the case, as unavoidable errors will arise in taking the bearings and measuring the distances.

(6.) But if the error is more than *one link for every five chains*, the survey is considered too inaccurate, and a resurvey should be made. When the errors in latitude and departure are less than this, they may be apportioned among the several latitudes and departures in the manner following, which is called

BALANCING THE WORK.

Let e = error in latitude or departure,
S = sum of latitudes or departures,
d = any latitude or departure,
c = correction for that latitude or departure;

then $$\frac{e}{S} \times d = c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Apply these corrections to their corresponding differences of latitudes and departures by adding to the less and subtracting from the greater; the latitudes and departures will then be properly balanced.

(7.) The survey being balanced, the next step is to make the column of *double meridian distance.*

1. *The double meridian distance of the first course is equal to its departure.*

2. *The double meridian distance of the second course is equal to the double meridian distance of the first course plus its departure, plus the departure of the second course.*

3. *The double meridian distance of any course is equal to the double meridian distance of the preceding course plus its departure, plus the departure of the course itself;* and,

4. *The double meridian distance of the last course (obtained as in* (3)) *is equal to its departure with an opposite sign. This is the second place of verification.*

⁂ It is to be understood that *plus* is used in its algebraic sense. East is considered *positive,* and west *negative* (Inst. of Alg., Def. 3).

Hence, if the double meridian distance and the departure are both east or both west, they must be added together; if one is east and the other west, the less must be taken from the greater, and the sign of the greater prefixed (Inst. of Alg., Art. 27).

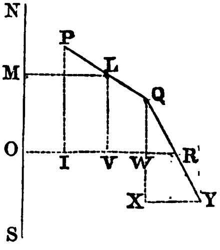
The assumed meridian may pass through any station; it is better, however, to make it pass through the most easterly or the most westerly station.

(8.) *Demonstration of the Rule for Double Meridian Distances.*

In the annexed diagram, let QY be any course, and PQ the preceding one; also, let R and L be their middle points.

Draw LM, RO, and XY perpendicular to the assumed meridian NS.

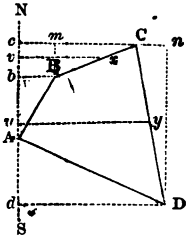
Draw, also, PI, LV, and QX parallel to NS. Then 2RO is the double meridian distance of QY, and 2LM =2OV is the double meridian distance of PQ.



Now $2RO = 2OV + 2VW + 2WR$;
but $2VW = IW =$ departure of PQ, and
$2WR = XY =$ departure of QY;
$\therefore 2OR = 2OV + IW + XY$,
whence the truth of the Rule is manifest.

Cor. If the meridian NS were drawn through Q, then QY would be the first course, 2OV and IW would be *zero*, and the double meridian distance would be equal to the departure.

(9.) *To determine the Area of the Survey.*



In the adjoining diagram, the double area of the triangle AbB is equal to A$b \times b$B. The double area of the trapezoid bcCB is equal to $2vx \times bc$ (Mens. of Sur., Prob. VI.). The double area of the triangle ADd is equal to dD $\times$ Ad, and the double area of the trapezoid dcCD is equal to $2uy \times n$D.

If from the double area of the trapezoid dcCD we subtract the sum of the areas of the triangles dAD, AbB, and the area of the trapezoid bcCB, the remainder will evidently be the double area of the tract ABCDA.

In examining the above, we find the latitude Ab to be north, the double meridian distance bB east: both positive; hence their product is *positive.*

The latitude bc is north, the double meridian distance $2vx$ east; hence their product is *positive.*

The latitude Ad is south, the double meridian distance Dd is west: both *negative;* hence their product is *positive.*

Finally, the latitude nD is south, the double meridian distance uy is east: the one *negative,* the other *positive;* hence their product is *negative.* From which we derive the following

RULE.

North latitude and east double meridian distance being called *positive,* south latitude and west double meridian distance being called *negative.* *Multiply each double meridian distance by its corresponding latitude,* *observing that like signs produce* plus, *and unlike signs* minus.

Take the sum of all the positive products, and also the sum of all the negative products.

The difference between these two sums will be the double area of the field or survey.

Note.—If the distances are measured in chains, the double area will be obtained in *square chains*, of which *ten make one acre;* or, if in rods, the double area will be in square rods, etc.

The above reasoning is applicable to any rectilinear figure, whatever may be the number and position of the sides. Hence the rule is general.

The preceding investigation will be better exemplified by an example.

No. 1. The following is a copy of a field-book made in surveying a farm belonging to W. Smart, Esq., of Flushing, L. I.:

Commencing at the N.E. corner of the farm, and
running S. 52° ¼ E. 1.73 chains,
thence S. 33° E. 25.07 "
" S. 36° E. 5.54 "
" S. 64° ¾ W. 11.72 "
" N. 67° W. 3.39 "
" N. 69° ¼ W. 2.70 "
" N. 64° ½ W. 0.79 "
" N. 41° W. 5.88 "
" N. 47° ½ E. 5.46 "
" N. 30° ½ W. 18.69 "
" N. 57° ¾ E. 3.22 "
" N. 59° ¾ E. 6.11 " to the place
of beginning.

We now form a table, the first *vertical* column of which is for the stations, 1, 2, 3, etc.; the second, for the bearings; the third, for the distances; the fourth and fifth, for the difference of latitude; the sixth and seventh, for the departures; the eighth and ninth, for the corrections. The tenth, eleventh, twelfth, and thirteenth contain the *corrected* latitudes and departures. The fourteenth is for the double meridian distance. The fifteenth and sixteenth for the double areas, *plus* and *minus*. Thus,

Sta.	Bearings.	Dist.	N. L.	S. L.	E. D.	W. D.	C.	C.	N. L.	S. L.	E. D.	W. D.	Double Mer. Dis.	+Area.	−Area.
1	S. $52\frac{1}{4}$° E.	1.73		1.06	1.37					1.06	1.37		−35.19	37.3014	
2	S. 33° E.	25.07		21.03	13.66		.02	.01		21.05	13.65		−20.17	424.5785	
3	S. 36° E.	5.54		4.49	3.26		.01			4.50	3.26		− 3.26	14.6700	
4	S. $64\frac{3}{4}$° W.	11.72		5.00		10.60	.01			5.01		10.60	−10.60	53.1060	
5	N. 67° W.	3.39	1.32			3.12			1.32			3.12	−24.32		32.1040
6	N. $69\frac{1}{4}$° W.	2.70	0.96			2.52			0.96			2.52	−29.96		28.7616
7	N. $64\frac{1}{2}$° W.	0.79	0.34			0.71			0.34			0.71	−33.19		11.2846
8	N. 41° W.	5.88	4.43			3.86	.01		4.42			3.86	−37.76		166.8992
9	N. $47\frac{1}{2}$° E.	5.46	3.69		4.03				3.69		4.03		−37.59		138.7071
10	N. $30\frac{1}{2}$° W.	18.69	16.10			9.49	.01	.01	16.09			9.50	−43.06		692.8354
11	N. $57\frac{3}{4}$° E.	3.22	1.72		2.73				1.72		2.73		−49.83		85.7076
12	N. $59\frac{3}{4}$° E.	6.11	3.08		5.27				3.08		5.27		−41.83		128.8364
			31.64	31.58	30.32	30.30	.06	.02	31.62	31.62	30.31	30.31		529.6559	1295.1359
			.58		.30										529.6559

Error in lat. = .06 .02 = Er. in dep.

Sum of lats. = 63.22, and sum of deps. = 60.62.

$\therefore$ Er. in lat. for 1 ch. $= \frac{e}{S} = \frac{.06}{63.22} = .0000095$;

$\therefore$ Er. in dep. " " $= \frac{e}{S} = \frac{.02}{60.62} = .0000033$.

20) 755.4800

A. 37.774

4

R. 3.096

40

P. 3.840

Multiplying each particular latitude by the above ratio for latitude, we find the correction for each latitude;

And multiplying each departure by the ratio for departure, we find the correction for each departure.

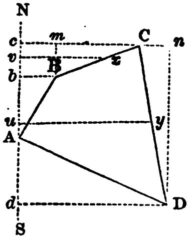
But, inasmuch as we do not use a quantity less than a link, we correct them as in the table.

It is as well, when the error is very small, to balance it by inspection.

Having calculated the area, we may proceed

(10.) *To make a Map of the Survey.*

There are several methods of protracting a survey, most of which will suggest themselves to the reader. Perhaps as simple and complete a method as any is the following, which belongs to the second example.



Having determined the scale on which the map is to be made, draw the line NS for a north and south line, which call the principal meridian. In this line assume any point, as A, for the first station. From the Traverse Table of *corrected* latitudes and departures, take the latitude of that station, and set it from A to *b*, because the latitude is north. At the point *b* erect a perpendicular, and upon that perpendicular set the departure from *b* to B (to the right of it is *plus*, to the left of it is *minus*); join AB, which will be the course and distance of the first side. From *b*, upon the principal meridian, set the latitude of the second station from *b* to *c*; erect a perpendicular at *c*, and lay thereon, first, the departure *b*B, then from that the departure of the second station to C; join BC, which will be the course and distance of the second side. From *c*, upon the principal meridian, set the latitude of the third station from *c* to *d*; at *d* erect a perpendicular, and lay thereon, first, the departure *c*C, then the departure of the third station to D; join CD, the course and distance of the third side. Again: from *d*, upon the principal meridian, lay off the latitude of the fourth station, which, as it is the last side, must fall upon the point A; the departure of the third and fourth stations must make up the line D*d*; join DA, and the map is finished.

We might have drawn meridians through the points B, C, D, and from them laid off the difference of latitude and departure of each station respectively; always remembering *that north latitude is to be set above and south latitude below the point, and that east departure is to be laid to the right and west departure to the left of the meridian.*

No. 2. Required the area and map of a piece of land of which the following are the field-notes:

1. N. 23° E. 17.00 chains, to a marked tree.
2. N. 83° E. 11.00 " to B. W.'s land.
3. S. 14° E. 23.00 " to the highway.
4. N. 77° W. 23.66 " to the place of beginning.

Ans. 32 A. 1 R. 18 P.

No. 3. Required the area and map of a tract of land lying in the town of Flushing, L. I., belonging to Peter Nostrand, Esq., surveyed by B. F. Willetts. The following are his field-notes:

1. S. 45° ¼ E. 41.22 chains,
2. S. 48° ½ E. 40.12 "
3. N. 50° ½ E. 9.66 "
4. N. 78° W. 0.62 "
5. N. 47° W. 83.27 "
6. N. 30° W. 1.52 "
7. S. 26° W. 4.18 "
8. S. 28° ½ W. 5.86 " to the place of beginning.

Ans. 81 A. 1 R. 37 P.

No. 4. Required the area and map of a tract of land of which the following are the field-notes:

1. N. 16° ½ E. 14.35 chains,
2. East 7.82 "
3. S. 3° ½ W. 14.45 "
4. N. 86° ½ W. 11.04 " to the place of beginning.

Ans. 13 A. 1 R. 17 P.

PROBLEM II.

(11.) *To supply Omissions in the Field-book.*

Given the bearings and distances of all the sides of a tract of land except the bearing and distance of one side, to find these, and thence the area.

Obtain the difference of latitude and departure of the given sides as in Prob. I., and what the northings and southings want of balancing will be the difference of latitude, and what the eastings and westings want of balancing will be the departure of the omitted side.

The bearing and distance of this side may then be found by (1) and (2). The area can then be found by Prob. I.

EXAMPLES.

1. In the following field-book, the bearing and distance of the fifth side are wanting.

Station.	Bearings.	Distance.	N. L.	S. L.	E. D.	W. D.	D. M. D.	+Area.	—Area.
1.	S. 45½° W.	15.16		10.62		10.81	+44.67		474.3954
2.	N. 50° W.	22.10	14.20			16.93	+16.93	240.4060	
3.	North.	18.83	18.83						
4.	N. 85° E.	35.65	3.11		35.52		+35.52	110.4672	
5.				(19.79)	(21.20)		+92.24		1825.4296
6.	S. 20½° W.	23.80		22.29		8.33	+105.11		2342.9019
7.	N. 51¼° W.	26.47	16.56			20.65	+76.13	1260.7128	
			52.70 32.91	32.91	35.52	56.72 35.52		1611.5860	4642.7269 1611.5860
		Diff. lat.=	19.79		Dep.=	21.20			2)3031.1469 1515.5573¼

Ans. 151 A., 2 R., 9 P.

Eq. (1.)	As diff. of lat. 19.79 ar. co.	8.703554
	: departure 21.20 log.	1.326336
	:: rad.	10.000000
	: tan. bearing 46° 58′	10.029890

Eq. (2.)	As cos. bearing ar. co.	0.165946
	: rad.	10.000000
	:: diff. of lat.	1.296446
	: dist. 29.00	1.462392

Hence the bearing and distance of the fifth side are
S. 46° 58′ E., dist. 29.00 chains.

By this method the bearing and distance between two places may be determined when the one can not be seen from the other.

Ex. 2. Given the bearings and distances between two places, as follows, viz.:

1. S. 10° E. 92.20 chains.
2. S. 15° W. 120.50 "
3. S. 18° ½ W. 205.00 "
4. S. 71° ½ E. 68.00 "

to find the bearing and distance of a straight road from the first to the fourth.

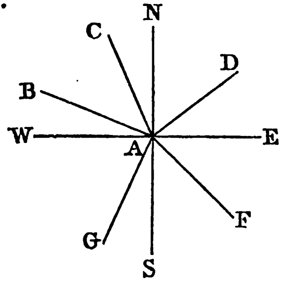
Ans. S. 2° 7′ 48″ W. 423.50 chains.

Ex. 3. Given, 1st, S. 40° ½ E. 31.80 ch.; 2d, *wanting;* 3d, N. 29° ¼ E. 2.21 ch.; 4th, N. 28° ¾ E. 35.35 ch.; 5th, N. 57° W. 21.10 ch.; 6th, S. 47° W. 31.30 ch., to find the area, and the bearing and distance of the 2d side.

Ans. { Area, 92 A. 3 R. 30 P.
2d Side, N. 56° 23′ E., 2.22 ch.

PROBLEM III.

(12.) *To find the angle included between any two given bearings.*



Let NS denote a meridian passing through A; and AC, AB, two bearings from A, both northwest. It is evident that the angle BAC is equal to NAB−NAC. Again, the angle CAD contained between the bearings AC and AD, the one being northwest, the other northeast, is equal to CAN+NAD.

Or, if AD is northeast and AF southeast, then the angle DAF contained between them is equal to 180°−(NAD+SAF).

And if AD is northeast and AG southwest, then the angle GAD contained by these lines is equal to 180°−(SAD−SAG).

Whence we derive the following

RULES.

1. *If the meridional letters are alike, and those of the departure are also alike, the difference between the bearings will be the angle contained between them.*

2. *If the meridional letters are alike, and those of the departure unlike, the sum of the bearings will be the angle contained between them.*

3. *If the meridional letters are unlike, and those of the departure alike, the angle contained between them will be equal to* 180° *minus the sum of the bearings.*

4. *If the meridional letters are unlike, and those of the departure also unlike, the angle contained between them will be equal to* 180° *minus the difference of the bearings.*

PROBLEM IV.

(13.) *To determine the Area of a Tract of Land bounded by Irregular Curves.*

When the boundaries of a tract of land are irregular, as is frequently the case where brooks or creeks form those boundaries, it would be troublesome to take all the bearings and distances requisite to obtain the area with accuracy. In cases of this kind, it is usual to run straight lines including the area nearly; and in taking these straight lines, perpendicular distances must be measured from them to each bend in the irregular boundary. These perpendicular distances are called offsets. The lengths of the offset, and the distance of the foot of each from the commencement of the straight line, should be carefully noted in the field-book; and whether the offsets are within or without the straight line should be particularly designated.

From these notes the area of the figure contained within the straight lines will be found by Problem I.; and the area or areas contained between the straight lines and the irregular boundary can be found in the manner following.

Subtract the distance of each offset from that of the one immediately following; the remainder will be the distance intercepted between each two adjacent offsets. Place these one under another in a column, as in the example annexed. Likewise take the sums of each two adjacent offsets and place them in the next column, so as to correspond with the intercepted distances.

Multiply the sum of each two adjacent offsets by their intercepted distances; then half the sum of the products will be the area of the offset on that line. Proceed in the same manner with all the others (Prob. VI., Mens. Surfaces).

When the straight lines are within the irregular boundaries of the field, add the areas of the offsets to the area inclosed by the straight lines; but when the straight lines are without the boundary, subtract the areas of the offsets.

The offsets divide the area contained between the straight line and the irregular boundary into trapezoids and triangles; hence the truth of the rule is manifest.

EXAMPLES.

Required the area of a piece of salt meadow, bounded on one side by a creek. The following are the field-notes:

1. N. 16° ½ E. 14.35 ch. to the causeway.
2. East 7.82 ch. to the east side of the creek.
3. S. 3° ½ W. 14.45 ch. to the line of offsets.
4. N. 86° ½ W. 11.04 ch. to the place of beginning.

Right-hand offsets on the third side:

	Distance.	Offsets.
1.	0.00 chains.	0.30 chains.
2.	0.95 “	0.84 “
3.	2.03 “	0.86 “
4.	3.28 “	0.50 “
5.	5.20 “	1.80 “
6.	7.43 “	2.35 “

	Distance.		Offsets.
7.	8.98 chains	. . .	1.45 chains.
8.	10.46 "	. . . . .	1.08 "
9.	11.71 "	. . . . .	1.85 "
10.	14.45 "	. . . . .	0.35 "

The area of the survey within the straight lines found by Prob. I. is 13 A. 1 R. 17.9 P.

To find the Area of the Offsets.

No.	Dist.	Offsets.	Inter. Distance.	Sums of Offsets.	Products.
1.	0.00	0.30			
2.	0.95	0.84	0.95	1.14	1.0830
3.	2.03	0.86	1.08	1.70	1.8360
4.	3.28	0.50	1.25	1.36	1.7000
5.	5.20	1.80	1.92	2.30	4.4160
6.	7.43	2.35	2.23	4.15	9.2545
7.	8.98	1.45	1.55	3.80	5.8900
8.	10.46	1.08	1.48	2.53	3.7444
9.	11.71	1.85	1.25	2.93	3.6625
10.	14.45	0.35	2.74	2.20	6.0280

2)37.6144

18.8072 sq. ch.

=1 A., 3 R., 20.9 P.

		A.	R.	P.
Area of the part within the straight lines	=	13	1	17
" " " " offsets	=	1	3	20
Whole area	=	11	1	37

LAYING OUT AND DIVIDING LAND.

PROBLEM V.

(14.) *To lay out a given quantity of Land in the form of a triangle or parallelogram, one side and an adjacent angle being given.*

FOR A TRIANGLE.

Let a be the given side, and C the given angle; then the *area of the triangle* $=\frac{1}{2}ab \sin. C$;

or, $ab \sin. C = 2 \text{ area}$;

$$\therefore b = \frac{2 \text{ area}}{a \sin. C}.$$

Or, log. b=log. 2 area+ar. co. log. a+ar. co. log. sin. C−10.

From which we derive the following

RULE.

To the logarithm of double the area, add the arithmetical complement of the logarithm of the given side, and the arithmetical complement of the logarithmic sine of the given angle, the sum, minus ten, will be the logarithm of the other adjacent side.

For a parallelogram, the same rule will apply by taking the *logarithm of the area.*

EXAMPLES.

1. Given one side of a triangle=6.15 chains, and an angle adjacent to this side=73° 20′, to cut off 3 acres: what will be the length of the other adjacent side?

Double area=60 log.	1.778151
a=6.15 ar. co. log. .	9.211125
C=73° 20′ ar. co. sin.	0.018639
Adjacent side=10.18 chs.	1.007915

2. Given one side of a parallelogram=32.26 chains, an angle adjacent to this side=83° 30′, to cut off 74 acres; find the other side adjacent to the given angle.

Area=740 log.	2.869232
a=32.26 ar. co. log. . .	8.491336
C=83° 30′ ar. co. sin. .	0.002801
Other side=23.09 chs.	1.363369

3. Given one side of a triangle=16.25 chains, an angle adjacent to this side=60°, and the area 10 acres, to find the other side adjacent to the given angle.

Ans. 14.20 chains.

4. Given one side of a parallelogram=30 rods, an angle adjacent to this side=71° 15′, and the area 4 acres, to find the other side adjacent to the given angle.

Ans. 22.53 rods.

PROBLEM VI.

(15.) *Given the bearings of the adjacent sides of a tract of land, to cut off a triangle containing a given area by a line running a given course.*

1°. From the given bearings of the sides, find the three angles of the required triangle (by Prob. III., Art. 12). Then, if A=area, we have by (Prob. III., Mens. of Surfaces),

$$\frac{a^2 \text{ sin. B sin. C}}{2 \text{ sin. A}} = \text{A}.$$

$$\therefore a^2 = \frac{2\text{A sin. A}}{\text{sin. B sin. C}};$$

or, $\text{log. } a = \frac{1}{2}\{\text{log. 2A} + \text{log. sin. A} + \text{ar. co. log. sin. B} + \text{ar. co. log. sin. C} - 10\}$.

From which we derive the following

RULE.

To the logarithm of the double area, add the logarithmic sine of the angle opposite the required side, and the arithmetical complements of the logarithmic sines of the other two angles; reject ten; half the remainder will be the logarithm of the required side.

EXAMPLES.

1. Given the bearing of one side N. 87° 30′ E., and at the same point the bearing of another side N. 27° 30′ E., to lay out a triangle containing 10 acres by a line from some point on the first side, a course N. 38° W., to find the distance on the first side.

We find the angle opposite the required side to be 65° 30′, and the other angles 60° and 54° 30′.

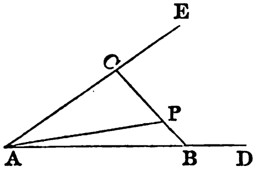
∴ double area=200 sq. chs. log. .	2.301030
A=65° 30′ sin.	9.959023
B=60° ar. co. sin. . .	0.062469
C=54° 30′ ar. co. sin.	0.089314
2 log. a=	2)2.411836
	1.205918

a=16.07 chains.

2. Given the bearing of one side of a tract of land S. 53° 15′ E., and the bearing of an adjacent side taken at the same point N. 55° E., to cut off 4 acres by a line running N. 4° W., to find the distance on the first side.
Ans. 9.76 chains.

PROBLEM VII.

(16.) *Given the bearings of two adjacent sides of a tract of land, and also the bearing and distance from the angular point to a point within these sides; to cut off a triangle containing a given area by a line passing through this point.*



Let AD and AE be the sides of the tract of land, P the given point, and BC the proposed line cutting off the area ABC=A.

Let $\quad AP=a,$
the angle $PAB=\theta, \quad PAC=\phi,$
and $\quad APC=\beta;$
therefore $\quad APB=\pi-\beta; \quad ACP=\pi-\phi-\beta;$
and $\quad ABP=\pi-\theta-(\pi-\beta)=\beta-\theta.$

Hence the area of the triangle

$$ACP=\frac{\frac{1}{2}a^2 \sin.\phi \sin.\beta}{\sin.(\beta+\phi)} \text{ (Mens. of Surf., Prob. III.)}$$
$$=\frac{\frac{1}{2}a^2}{\cot.\phi+\cot.\beta}.$$

$$\text{Area of } \Delta \text{ ABP}=\frac{\frac{1}{2}a^2 \sin.\theta \sin.\beta}{\sin.(\beta-\theta)};$$
$$=\frac{\frac{1}{2}a^2}{\cot.\theta-\cot.\beta}.$$

But these two triangles together constitute the given magnitude ABC.

$$\therefore \frac{\frac{1}{2}a^2}{\cot.\phi+\cot.\beta}+\frac{\frac{1}{2}a^2}{\cot.\theta-\cot.\beta}=A,$$

which gives

$$\cot.^2\beta-(\cot.\theta-\cot.\phi)\cot.\beta=$$
$$\cot.\phi \cot.\theta-\frac{a^2}{2A}(\cot.\theta+\cot.\phi) \quad . \quad . \quad . \quad (a)$$
$$\therefore \cot.\beta=\tfrac{1}{2}(\cot.\theta-\cot.\phi)\pm$$
$$\tfrac{1}{2}\left\{(\cot.\theta+\cot.\phi)^2-\frac{2a^2}{A}(\cot.\theta+\cot.\phi)\right\}^{\frac{1}{2}},$$

or
$$\cot.\beta=\tfrac{1}{2}(\cot.\theta-\cot.\phi)\pm$$
$$\tfrac{1}{2}(\cot.\theta+\cot.\phi)\left[1-\frac{2a^2}{\mathrm{A}}\left(\frac{1}{\cot.\theta+\cot.\phi}\right)\right]^{\frac{1}{2}} \quad . \quad (b)$$

Now find an angle ω, such that
$$\sin.^2\omega=\frac{2a^2}{\mathrm{A}}\cdot\frac{1}{\cot.\theta+\cot.\phi}=\frac{2a^2}{\mathrm{A}}\cdot\frac{\sin.\theta\sin.\phi}{\sin.(\theta+\phi)} \quad . \quad (c)$$
which being substituted in (Eq. b), gives
$$\cot.\beta=\tfrac{1}{2}(\cot.\theta-\cot.\phi)\pm\tfrac{1}{2}(\cot.\theta+\cot.\phi)\cos.\omega. \quad (d)$$

But $\mathrm{AB}=\dfrac{a\sin.\beta}{\sin.(\beta-\theta)}=\dfrac{a}{\sin.\theta(\cot.\theta-\cot.\beta)}$,
or, by substituting for $\cot.\beta$ its value as above, we have
$$\mathrm{AB}=\frac{a}{\tfrac{1}{2}\sin.\theta(\cot.\theta+\cot.\phi)(1\pm\cos.\omega)}$$
$$=\frac{2a\sin.\phi}{\sin.(\theta+\phi)(1\pm\cos.\omega)}.$$

But $1+\cos.\omega=2\cos.^2\tfrac{1}{2}\omega$, and $1-\cos.\omega=2\sin.^2\tfrac{1}{2}\omega$;
$$\therefore\mathrm{AB}=\frac{a\sin.\phi}{\sin.(\theta+\phi)\cos.^2\tfrac{1}{2}\omega},\text{ or }\frac{a\sin.\phi}{\sin.(\theta+\phi)\sin.^2\tfrac{1}{2}\omega} \quad (e)$$
In like manner we find
$$\mathrm{AC}=\frac{a\sin.\theta}{\sin.(\theta+\phi)\cos.^2\tfrac{1}{2}\omega},\text{ or }\frac{a\sin.\theta}{\sin.(\theta+\phi)\sin.^2\tfrac{1}{2}\omega} \quad (f)$$
In logarithms, (Eq. c) becomes
$$2\log.\sin.\omega=\log.2+2\log.a+\log.\sin.\theta+\log.\sin.\phi+$$
$$\text{ar. co. log. A}+\text{ar. co. log. sin.}(\theta+\phi)-10 \quad . \quad (g)$$
(Eq. e) becomes
$$\log.\mathrm{AB}=\log.a+\log.\sin.\phi+\text{ar. co. log. sin.}(\theta+\phi)+$$
$$\text{ar. co. }2\log.\cos.\tfrac{1}{2}\omega-10 \quad . \; . \; . \; . \; . \quad (h)$$
or
$$\log.\mathrm{AB}=\log.a+\log.\sin.\phi+\text{ar. co. log. sin.}(\theta+\phi)+$$
$$\text{ar. co. }2\log.\sin.\tfrac{1}{2}\omega-10 \quad . \; . \; . \; . \quad (i)$$

Hence there are always two lines that can be drawn through the given point, cutting off the same area except when $\omega=90°$.

EXAMPLES.

1. Given the bearings of two sides of a triangular piece of land taken at the same point, N. 82° E., and S. 50° 30′ E.; also the bearing and distance of a spring from the same point S. 78° E. 11.78 chains, to cut off 6 A. 1 R. 38 P. by a line passing through the spring

and intersecting the two sides of the triangle. Find the distance on either side.

The diagram can easily be constructed.

By Art. 12, Prob. III.:

$$\angle PAC = 20° = \phi;$$
$$\angle PAB = 27°\ 30' = \theta;$$

and $\therefore \theta+\phi = 47°\ 30'$; $a = 11.78$, and $A = 64.875$.

Find ω by (Eq. g).

log. $2 =$	0.301030
$a = 11.78$ twice log. $=$	2.142290
$\theta = 27°\ 30'$ sin. $=$	9.664406
$\phi = 20°$ sin. $=$	9.534052
$A = 64.875$ ar. co. $=$	8.187923
$(\theta+\phi) = 47°\ 30'$ ar. co. sin. $=$	0.132369
twice log. sin. $\omega =$	19.962070
log. sin. $\omega =$	9.981035

$$\therefore \omega = 73°\ 12',$$
$$\tfrac{1}{2}\omega = 36°\ 36'.$$

Find AB by (h) or (i).

$a = 11.78$ log.	1.071145
$\phi = 20°$ sin.	9.534052
$(\theta+\phi) = 47°\ 30'$ ar. co. sin.	0.132369
$\tfrac{1}{2}\omega = 36°\ 36'$ ar. co. 2 log. cos.	0.190766
AB $= 8.48$ chs.	0.928332

Then

$a = 11.78$ log.	1.071145
$\phi = 20°$ sin.	9.534052
$(\theta+\phi) = 47°\ 30'$ ar. co. sin.	0.132369
$\tfrac{1}{2}\omega = 36°\ 36'$ ar. co. 2 log. sin.	0.449180
AB $= 15.37$ log.	1.186746

In like manner, find AC $= 11.46$, or 20.73.

2. Given the bearing of AC $=$ N. 12° E., the bearing of AB, N. 78° 30′ E., and the bearing and distance of AP, N. 45° 15′ E., 13.16 chains, to cut off 11 A., 2 R., 11.2 P. by a line passing through P, and intersecting AC and AB. Find AB and AC.

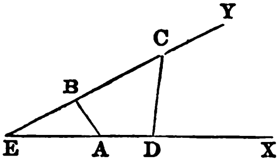
Ans. AB $= 18.23$ or 13.84 chains.
AC $= 13.84$ or 18.23 "

PROBLEM VIII.

(17.) *The bearings of three adjacent sides of a tract of land and the length of the middle one being given, to cut off a given area by a line running a given course.*

Solution.

1°. When the sum of the angles A and B is greater than 180°.



Let XA, AB, BY be the three sides; CD the division line required.

Produce XA, YB, to meet in the point E.

Put the angle AEB$=\theta$, EAB$=\gamma$, ABE$=\omega$, EDC$=\beta$, and DCE$=\phi$, all of which are given.

Let AB$=a$, CD$=x$, and the given area$=$A.

Then the area of the triangle EAB$=\dfrac{\frac{1}{2}a^2 \sin.\omega \sin.\gamma}{\sin.\theta}$,

and the area of the triangle EDC $=\dfrac{\frac{1}{2}x^2 \sin.\phi \sin.\beta}{\sin.\theta}$.

It is evident that the given area must be the difference of these two areas.

If DC passes as in the above figure, then

$$\frac{\frac{1}{2}x^2 \sin.\phi \sin.\beta}{\sin.\theta}-\frac{\frac{1}{2}a^2 \sin.\omega \sin.\gamma}{\sin.\theta}=\text{A},$$

or $$x^2 \sin.\phi \sin.\beta-a^2 \sin.\omega \sin.\gamma=2\text{A} \sin.\theta.$$

Whence $$x=\sqrt{\frac{a^2 \sin.\omega \sin.\gamma}{\sin.\phi \sin.\beta}\left[1+\frac{2\text{A} \sin.\theta}{a^2 \sin.\omega \sin.\gamma}\right]}.$$

Put $$\tan.^2 \psi=\frac{2\text{A} \sin.\theta}{a^2 \sin.\omega \sin.\gamma} \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

Then $$x=a \sec.\psi\sqrt{\frac{\sin.\omega \sin.\gamma}{\sin.\phi \sin.\beta}}.$$

And the distance AD$=$ED$-$EA.

But $$\text{ED}=\frac{x \sin.\phi}{\sin.\theta}, \text{ and EA}=\frac{a \sin.\omega}{\sin.\theta}.$$

$$\therefore \text{AD}=\frac{x \sin.\phi-a \sin.\omega}{\sin.\theta}.$$

Substitute for x its value, $a \sec.\psi\sqrt{\dfrac{\sin.\omega \sin.\gamma}{\sin.\phi \sin.\beta}}$, and we have

$$AD=\frac{a\sin.\phi\sec.\psi\sqrt{\frac{\sin.\omega\sin.\gamma}{\sin.\phi\sin.\beta}}-a\sin.\omega}{\sin.\theta};$$

which, by a little reduction, becomes

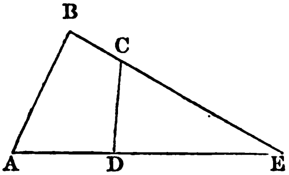
$$AD=\frac{a\sin.\omega}{\sin.\theta}\left[\sec.\psi\sqrt{\frac{\sin.\phi\sin.\gamma}{\sin.\omega\sin.\beta}}-1\right].$$

Put $\sec.\psi\sqrt{\frac{\sin.\phi\sin.\gamma}{\sin.\omega\sin.\beta}}=\sec.^2\delta$ . . . . . . (B)

Then $AD=\frac{a\sin.\omega}{\sin.\theta}(\sec.^2\delta-1)=\frac{a\sin.\omega\tan.^2\delta}{\sin.\theta}$. (C)

2°. When the sum of the angles A and B is less than 180°, we have the equation

$$\frac{\frac{1}{2}a^2\sin.\omega\sin.\gamma}{\sin.\theta}-\frac{\frac{1}{2}x^2\sin.\phi\sin.\beta}{\sin.\theta}=A.$$



Whence $x=\sqrt{\frac{a^2\sin.\omega\sin.\gamma}{\sin.\phi\sin.\beta}\left[1-\frac{2A\sin.\theta}{a^2\sin.\omega\sin.\gamma}\right]}.$

Put $\sin.^2\psi=\frac{2A\sin.\theta}{a^2\sin.\omega\sin.\gamma}$ . . . . . . (D)

Then $x=a\cos.\psi\sqrt{\frac{\sin.\omega\sin.\gamma}{\sin.\phi\sin.\beta}}$,

and the distance AD=EA−ED; therefore

$$AD=\frac{a\sin.\omega-x\sin.\phi}{\sin.\theta}$$

$$=\frac{a\sin.\omega}{\sin.\theta}\left[1-\cos.\psi\sqrt{\frac{\sin.\phi\sin.\gamma}{\sin.\omega\sin.\beta}}\right].$$

Put $\sqrt{\frac{\sin.\phi\sin.\gamma}{\sin.\omega\sin.\beta}}=\sec.\delta$ . . . . . (E)

Then AD=

$2a\sin.\omega\,\mathrm{cosec.}\,\theta\sec.\delta\sin.\frac{1}{2}(\psi+\delta)\sin.\frac{1}{2}(\psi-\delta)$. (F)

When the division line runs parallel to AB, then

$AD=\frac{2a\sin.\omega\sin.^2\frac{1}{2}\psi}{\sin.\theta}$ . . . . . (G)

When $A>\frac{\frac{1}{2}a^2\sin.\omega\sin.\gamma}{\sin.\theta}$, in this 2d case the question is impossible.

EXAMPLES.

1. Given the bearings of three adjacent sides of a tract of land and the length of the middle one, as follows, viz.: 1st, N. 74° 45′ W.; 2d, N. 37° E., dist. 17.24 chs.; 3d, N. 84° E., to cut off twenty acres by a line running S. 20° W.; to find the distance on the first side.

Solution. The angles $A=\gamma=68° 15'$, $133°=B=\omega$, $64°=C=\phi$, $94° 45'=D=\beta$, and $21° 15'=E=\theta$.

$\therefore A+B>180°$, and we use the formulas of the first case.

To find ψ by (Formula A):

$2A=400$ log.	2.602060
$\theta=21° 15'$ sin.	9.559234
$a^2=(17.24)^2$ ar. co. twice log.	7.526926
$\omega=133°$ cosec.	10.135873
$\gamma=68° 15'$ cosec.	10.032073
	2)19.856166
$\psi=40° 17'$ tan.	9.928083

To find δ by (B):

$\phi=64°$ sin.	9.953660
$\gamma=68° 15'$ sin.	9.967927
$\omega=133°$ cosec.	10.135873
$\beta=94° 45$ cosec.	10.001494
$\psi=40° 17'$ twice sec.	10.235114
	4)40.294068
$\delta=32° 24'$ sec.	10.073517

To find AD by (C):

$a=17.24$ log.	1.236537
$\omega=133°$ sin.	9.864127
$\delta=32° 24'$ twice tan.	9.605026
$\theta=21° 15'$ cosec.	10.440766
$AD=14.01$ log.	1.146456

2. Given the bearings of three adjacent sides of a tract of land, and the length of the middle one, as follows, viz.: 1st, N. 31° 15′ W.; 2d, N. 58° 45′ E., dist. 13.50 ch.; 3d, S. 14° 45′ E., to cut off 8 acres by a line running N. 87° 30′ W. Required the distance on the first side.

Solution. The angles $A=\gamma=90°$, $B=\omega=73° 30'$, $C=\phi=72° 45'$, $D=\beta=123° 45'$, $E=\theta=16° 30'$.

$\therefore A+B<180°$.

To find ψ by (Formula D):

$2A=160$ log.	2.204120
$\theta=16° 30'$ sin.	9.453342
$a^2=13° 50'$ ar. co.	8.869666
$13° 50'$ ar. co.	8.869666
$\omega=73° 30'$ cosec.	10.018263
$\gamma=90°$ cosec.	10.000000
	2)19.415057
$\psi=30° 40'$ sin.	9.707528

To find δ by (E):

$\phi=72° 45'$ sin.	9.980012
$\gamma=90°$ sin.	10.000000
$\omega=73° 30'$ cosec.	10.018263
$\beta=123° 45'$ cosec.	10.080154
	2)20.078429
$\delta=23° 59'$ sec.	10.039214

$\frac{1}{2}(\psi+\delta)=27° 15' 30''$; $\frac{1}{2}(\psi-\delta)=3° 20' 30''$.

To find AD by (F):

$2a=27$ log.	1.431364
$\omega=73° 30'$ sin.	9.981737
$\theta=16° 30'$ cosec.	10.546658
$\delta=23° 59'$ sec.	10.039214
$\frac{1}{2}(\psi+\delta)=27° 15' 30''$ sin.	9.660868
$\frac{1}{2}(\psi-\delta)=3° 20' 30''$ sin.	8.765592
AD=2.663 log.	0.425433

3. Given the same as Quest. (2), to cut off 8 acres by a line running parallel to AB.

Solution. By (Formula G):

$2a=27$ log.	1.431364
$\omega=73° 30'$ sin.	9.981737
$\frac{1}{2}\psi=15° 21'$ twice sin.	8.844636
$\theta=16° 30'$ cosec.	10.546658
AD=6.37 log.	0.804395

4. Given the bearings of three sides of a tract of land and the length of the middle one, as follows, viz.: 1st, N. 80° 30′ W.; 2d, N. dist. 12 ch.; 3d, N. 58° E., to cut off 10 acres by a line running S. 14° 30′ E. Required the distance on the first side. *Ans.* AD=9.16 chains.

5. Given the bearings of three adjacent sides of a tract of land and the length of the middle one, as follows, viz.:

1st, East; 2d, N. 10° E., dist. 15 ch.; 3d, S. 58° 30′ W., to cut off 10 acres by a line running parallel to the second side. Required the distance on the first side.

Ans. AD=8.42 chains.

PROBLEM IX.

(18.) *Given the boundaries of a tract of land, to divide it into two equal parts by a line running from a given station and falling on the opposite side.*

1°. *Calculate the area of the whole tract* by Prob. I.

2°. *Find the area of the part included within the sides from the given station to the side on which the division line is to fall, and also the bearing and distance of the closing line.*

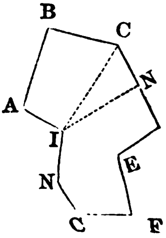
3°. *With this bearing, and the bearing of the line on which the division line is to fall, find the included angle.*

4°. *Subtract the area last found from the half area of the tract to be divided, and the remainder will be the area of a triangle to complete the division: the distance on the side to include this will be found* by Prob. V.

EXAMPLES.

1. Given the field-notes of a farm, as follows, to divide it into two equal parts by a line running from the 9th station and falling on the 3d side.

1.	N. 19°	E.	27.00	chains.
2.	S. 77°	E.	22.75	"
3.	S. 27°	E.	28.75	"
4.	S. 52°	W.	14.50	"
5.	S. 15° ½	E.	19.00	"
6.	West		17.72	"
7.	N. 36°	W.	11.75	"
8.	North		16.07	"
9.	N. 62°	W.	14.88	"



1°. The whole area calculated by Prob. I. =1525.398 sq. chs.

Half area =762.699.

2°. Area of IABCI=547.2379;

∴ area required to complete the division =215.4611.

Bearing and distance of the closing line, S. 33° 8′ W., 32.66 chs.

$\therefore \angle ICN = 60° 8'$.

By Prob. V., log. CN = log. 430.9222 + ar. co. log. 32.66 + ar. co. log. sin. 60° 8′;

$\therefore$ CN = 15.21 chains.

(19.) Problem IX. may be applied to divide a tract of land into any number of parts, either equal or unequal.

2. Given the boundaries of a tract of land, as follows, viz.:

1. S. 35° $\frac{1}{4}$ W. 11.20 chains.
2. N. 45° W. 24.36 "
3. N. 15° $\frac{1}{2}$ E. 10.80 "
4. S. 77° E. 16.00 "
5. N. 87° $\frac{1}{2}$ E. 21.50 "
6. S. 60° E. 14.80 "
7. South 10.91 "
8. N. 85° W. 29.28 " to the place of

beginning, to divide the tract into two equal parts by a line running from the 1st station and falling on one of the opposite sides. Required the bearing and distance of the division line and the distance on the opposite side.

Ans.
Bearing of division line N. 7° 18′ E.
Distance . . . . . . 15.28 ch.
Distance on 5th side . . 7.12 ch.

3. Given the courses and distances as in the above, to divide the tract into three equal parts by a line running from the 1st station and falling on the 4th side, and by a line from the 6th station falling on the 8th side.

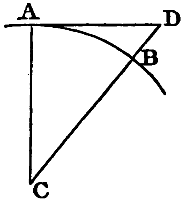
Ans.
Bearing of 1st division line N. 45° 41′ W.
Distance . . . . 25.64 ch.
Bearing of 2d division line S. 37° 39′ W.
Distance . . . . 20.35 ch.

LEVELING.

(20.) Two or more points are said to be on a level when they are equally distant from the centre of the earth or from the surface of a tranquil fluid.

A level surface, therefore, is one that is every where perpendicular to a plumb-line, or to the radius of the earth considered as a sphere.

This is called a *true level*, while a straight line or plane



that is perpendicular to the radius of the earth, or to a plumb-line at one point only, is denominated an *apparent level.*

Thus AD represents an apparent level, AB a true level, and BD a deviation of the one from the other, or the *difference* of level of the points A and B, referred to a tangent at A.

Knowing the tangent AD, we readily find BD from the equation

$$DB \times (2BC + BD) = \overline{AD}^2 \text{ (Geom., B. II., th. 21, Cor. 1),}$$

or
$$DB = \frac{AD^2}{2BC + BD} \quad \ldots\ldots\ldots \quad (a)$$

But as BD is always small in comparison with 2BC (the diameter of the earth), it may be neglected in the second member of (Eq. *a*); and in *most cases*, for the same reason, AB may be considered as equal to AD. Therefore

$$DB = \frac{AB^2}{2BC} \quad \ldots\ldots\ldots \quad (b)$$

In like manner, for another distance, AB′, we shall have

$$D'B' = \frac{\overline{AB'}^2}{2BC} \quad \ldots\ldots\ldots \quad (c)$$

Dividing equation (*b*) by (*c*), we obtain

$$\frac{DB}{D'B'} = \frac{\overline{AB}^2}{\overline{AB'}^2} \quad \ldots\ldots\ldots \quad (d)$$

That is, *the difference of level for different distances is as the square of those distances;*

or,
$$\overline{AB}^2 : \overline{AB'}^2 :: BD : B'D'.$$

The distance AD being supposed, for example, equal to *one statute mile*, or 5280 feet, and 2BC, the diameter of the earth, equal to 7920 miles, or 7920 × 5280, we have

$$BD = \frac{(5280)^2}{7920 \times 5280} = \frac{5280}{7920};$$

or, $\log. BD = \log. 5280 + \text{ar. co. } \log. 7920 - 10.$

That is,

5280 log. 3.722634
7920 ar. co. log. 6.101275

BD=0.6667 feet $\bar{1}.823909$
=8 inches.

Thus *the difference between the apparent and true level, answering to a distance of one mile, is* 8 *inches.*

(21.) For any other distance, as 2½ miles, for instance, we can, instead of repeating the above process, find the difference thus:

$$1^2 : (2.5)^2 :: \overset{\text{in.}}{8} : \overset{\text{in.}}{50}.$$

Taking BD=8 inches, and the difference of level being as the square of the distance, we have the following convenient rule for finding the difference of level, viz.:

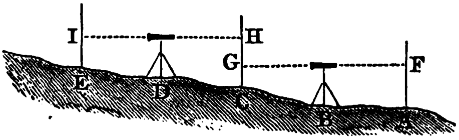
Two thirds of the square of the distance in miles are equal to the difference of level in feet.

To determine the Difference of Level between any two Points.

Place the leveling staves at each point perpendicular to the horizon, and at the middle point between them place the instrument properly adjusted. Direct the telescope to one of the staves, and order the person who holds the staff to slide up the vane until the centre line coincides with the centre of the telescope, and note the height of this line from the ground. Direct the telescope to the other staff, and perform the same operation. Level in a manner entirely similar from the second to the third, from the third to the fourth, etc., until you have arrived at the point proposed. Then the difference between the sum of the fore sights and that of the back sights will be the difference of level between the first and the last.

The following diagram will give the student a good idea of the process.

AF and CH are the back sights, and CG and EI the fore sights.



Before going to the field with the level, rule three columns, and head them *Stations*, *Back Sights*, *Fore Sights*, for the

Field-book.

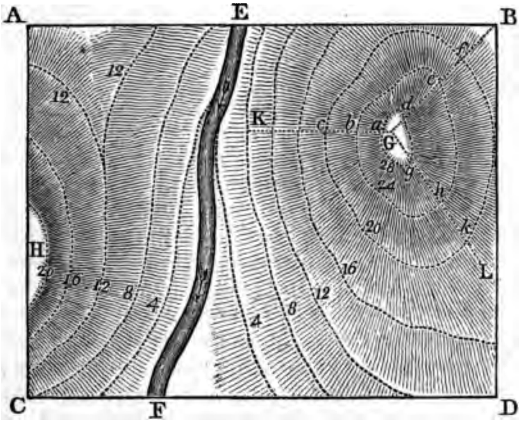
Stations.	Back Sights.	Fore Sights.
1	9	5
2	10	6
Sums........	19	11
Difference of level, 8 feet.		

TOPOGRAPHY.

(22.) The object of topography is to determine and designate on a map all the undulations in the surface of a particular tract of land. A map in which these undulations, the direction of streams, the position and extent of forests, etc., are designated, is called a *Topographical Map.*

After having made the survey, calculated the area, and drawn the map, then, if we suppose the surface to be intersected by a number of horizontal planes at equal distances from each other, and transfer all the lines of level in which these planes meet the surface of the ground to an assumed horizontal plane passing through the lowest point, making them occupy positions on that plane corresponding with their positions on their respective planes, the variations in the distances of the lines from each other, when thus transferred, will indicate the variations in the undulations of the ground. For, as the difference of level from line to line is the same, it is evident that the horizontal distances of the lines, taken in any direction, will *diminish* as the inclination in that direction *increases.*

Thus, in the diagram, which represents a tract of land, of which AB is the length, the breadth BD, the lines 4, 8, 12, 16 represent lines of level at the distance of 4 feet from each other, intersecting the surface of the ground. The lowest level passes through the point F, at which the stream EF leaves the tract. From that point the ground rises more rapidly to the left than it does to the right, as is shown by the lines of level being nearer to



each other on that side than on the other. In passing from F toward the corner B of the tract, we shall observe that the acclivity, which is at first gentle, increases till we come to the 16-feet level, then diminishes to the 20-feet level; it then increases, and more rapidly to the 24-feet level, and slightly diminishes to the 28-feet, or highest level. In descending, the declivity continually diminishes to B. From the point at which the shading begins above H, the ground descends moderately toward the corner A, the declivity diminishing till the surface becomes nearly level; and from the stream toward A, the acclivity, which is slight, diminishes, so that between the two branches of the 12-feet level and the corner A the ground is nearly level.

The distance between the assumed planes will depend on the extent of the survey, the inequalities on the surface, and the degree of exactness required in the delineation of the map. It may vary from 3 or 4 to 15 or 20 feet, according to circumstances.

The leveling required for a topographical map may be done either with a level or theodolite.

To make the Examinations and Measurements on the Field.

(23.) After having gone over the tract and noted the prominent features, select the highest point, as at G or H, and drive on it a stake. Place the theodolite or level directly over this stake, noting the height of the instrument. Direct the telescope to some well-defined mark at a distance, and send the assistant with the leveling staff down the hill, keeping him always in the same line. At a convenient distance, which will depend upon the abruptness of the slope, let the person with the staff slide up the vane until the horizontal spider-line intersects that on the vane. Measure the distance between these two points by carrying the chain horizontally; also note the height of the vane. The leveling instrument must now be removed and placed at this point, the assistant sent ahead to a point at which the horizontal spider-line intersects that on the vane at the same height as before. Continue this operation as far as necessary.

Other lines may be run in the same manner from the starting-point G or H, driving stakes at those points which have the same difference of level, accurately measuring the *horizontal distances* between them. If we take the bearings of these several lines they may be delineated on paper, and the intercepted distances, G*a*, *ab*, etc., G*d*, *de*, etc., being measured off on those lines, and curves being drawn through the points *a*, *d*, *g*, *b*, *e*, *h*, etc., will show the contour of the hill.

The map must then be shaded by drawing fine lines *perpendicular to the horizontal curves*, as in the diagram. The more abrupt the slope is, the more nearly together will be the horizontal curves.

A profile of either of the lines from G to K, or from G to B, etc., may be plotted thus:

Draw a horizontal line KG′, and assume G′. At this point erect a perpendicular, G′G, and make it equal to the height of G taken from the field-book. Lay off from G′ toward *k* the several measured distances G*a*, *ab*, *bc*, etc.; erect perpendiculars, and make them equal to the heights of those respective stations taken from the field-book. The curved line passing through the extremities of these perpendiculars will represent the profile of the surface of the ground along that line.

(24.) *The bearings, distances, and elevations of a route being given, to find the bearing, distance, and elevation of a straight line connecting the two places.*

By referring the points to three rectangular planes we readily obtain the formulas

Altitude =dist. × sin. elevation.
Diff. of lat.=dist. × cos. elevation × cos. bearing.
Departure =dist. × cos. elevation × sin. bearing.

Then the altitude of the last point above the first is the algebraic sum of the several altitudes; the difference of latitudes and departures are the algebraic sums of the differences of latitudes and departures of the several distances; and their bearing, elevation, and distance are calculated from the converse formulas

$$\text{tan. bearing} = \frac{\text{dep.}}{\text{diff. lat.}};$$

$$\text{tan. elevation} = \frac{\text{alt.} \times \text{cos. bearing}}{\text{diff. lat.}};$$

$$\text{distance} = \frac{\text{alt.}}{\text{sin. elevation}}.$$

EXAMPLES.

1. Given the following bearings, distances, and elevations of a route, viz.:

	Bearings.	Dist.	Elevation.
1.	N. 78° W.	30.50 ch.	+10°
2.	N. 81° W.	160.00 ch.	+5°
3.	N. 89° 45′ W.	50.00 ch.	+3° 20′
4.	S. 89° W.	67.25 ch.	+7° 15′
5.	S. 87° 30′ W.	100.75 ch.	−15° 45′
6.	S. 88° 30′ W.	150.50 ch.	−2° 10′,

to find the bearing, distance, and elevation of the straight line which connects the first and last station.

From the above formulas we have

log. alt.=log. dist.+log. sin. elevation−10.
log. difference of latitude=
log. dist.+log. cos. elev.+log. cos. bear.−20.
log. dep.=log. dist.+log. cos. elev.+log. sin. bear.−20.

Applying these:

Distance =30.50 log. 1.484300
Elevation=10° log. sin. 9.239670
Altitude 5.29 =0.723970

Distance log. 1.484300
Elevation 10° log. cos. 9.993351
Bearing 78° cos. 9.317879
Diff. of latitude=6.24 =0.795530

Distance, log. 1.484300
Elevation, cos. 9.993351
Bearing, sin. 9.990404
Departure=29.38 1.468055

Hence the altitude of the second station above the first is 5.29 chains, its difference of latitude is 6.24 chains, and departure 29.38 chains.

In like manner we obtain for the successive stations,
3d and 2d. alt.= 2.90 diff. lat.= 0.22 dep.= 49.92.
4th and 3d. alt.= 8.48 diff. lat.= −1.16 dep.= 66.70.
5th and 4th. alt.= −27.35 diff. lat.= −4.23 dep.= 96.88.
6th and 5th. alt.= − 5.69 diff. lat.= −3.93 dep.= 150.34.

Place these in their proper columns as in the annexed table, and take the difference between the sum of the positive and that of the negative quantities in the columns of difference of latitude, departure, and elevation.

Bearings.	Dist.	Elevation.	Altitude. chs.	Diff. Lat.	Departure.
1. N. 78° W.	30.50	+10°	+ 5.29	6.24 N.	29.38 W.
2. N. 81° W.	160.00	+ 5°	+13.94	24.93 N.	157.42 W.
3. N. 89¾° W.	50.00	+ 3° 20′	+ 2,90	.22 N.	49.92 W.
4. S. 89° W.	67.25	+ 7° 15′	+ 8.48	−1.16 S.	66.70 W.
5. S. 87½ W.	100.75	−15° 45′	−27.35	−4.23 S.	96.88 W.
6. S. 88½ W.	150.50	− 2° 10′	− 5.69	−3.93 S.	150.34 W.
N. 87° 42′ W.	556.90	− 0° 15′	− 2.43	22.07 N.	550.64 W.

The difference of altitude is −2.43; diff. of lat.=22.07; dep.=550.64.

Hence dep.=550.64 log. 2.74087
diff. lat.=22.07 1.34380
tan. bearing=87° 42′= 11.39707

alt. = 2.43 log.	0.38561
bearing = 87° 42′ cos.	8.60349
diff. lat. = 22.07 ar. co.	8.65620
elevation = 0° 15′ tan.	7.64530

alt. = 2.43 log.	0.38561
elevation = 0° 15′ cosec.	12.36018
distance = 556.90 =	2.74579

2. Given the following bearings, distances, and elevations, viz.:

1. N. 10° 15′ E., dist. 27.54 chs., elevation + 17° 54′
2. N. 28° 40′ W. " 100.00 " " + 20° 19′
3. N. 20° W. " 15.00 " " + 7° 43′
4. N. 20° W. " 37.26 " " − 5° 26′
5. N. 36° 17′ E. " 68.75 " " − 11° 13′,

to find the bearing, distance, and elevation of a line drawn from the beginning of the first to the end of the fifth line.

Ans. N. 4° 54½′ W., 213.74 chs., Elevation 7° 36⅓′.

NAVIGATION.

PLANE SAILING.

(25.) *Plane sailing* is the art of navigating a ship upon principles derived from the supposition that the earth is an extended plane. Upon this hypothesis, the *meridians*, which are semicircumferences of great circles passing from pole to pole, become parallel to each other.

1. The *course** of the vessel is the angle it makes with the meridian, being reckoned in points, etc., of the mariner's compass, or in degrees.

Distance is the path the ship has made in a direct course in a given time.

The *axis of the earth* is an imaginary straight line passing through its centre, around which it performs its daily revolution.

2. The *poles of the earth* are the extremities of this axis; the one to the north being called the *north pole*, the other the *south pole.*

3. The *latitude* of a place is the arc of the meridian comprehended between the place and the equator.

4. The *equator* is an imaginary great circle equally distant from the *north* and *south pole.*

5. Latitude is reckoned north and south of this great circle from 0° to 90°.

6. *Parallels of latitude* are small circles drawn parallel to the equator.

7. The *difference of latitude* of two places is the arc of a meridian intercepted between the parallels of latitude in which they are respectively situated. Hence the difference of latitude of two places on the same side of the equator is equal to the difference of their latitudes; and the difference of latitude of two places on opposite sides of the equator is equal to the sum of their latitudes.

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* A ship is said to continue on the same course when she cuts every meridian at the same angle. This course is called a *rhumb line.*

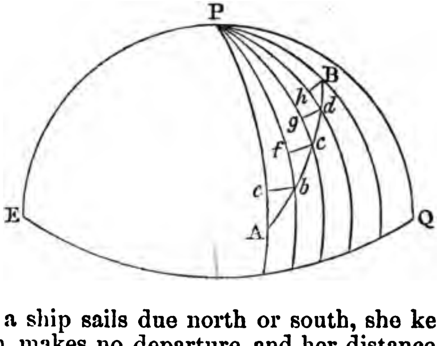
8. The *longitude* of a place is the arc of the equator intercepted between the first meridian and the meridian passing through the place.

9. The *first meridian* is arbitrarily assumed. That which passes through Greenwich, in England, is, however, the one most generally used by navigators.

10. The *difference of longitude* of two places is the arc of the equator intercepted between the meridians passing through those places. Hence the difference of longitude of two places on the same side of the first meridian is equal to the difference of their longitudes; and the difference of longitude of two places on opposite sides of the first meridian is equal to the sum of their longitudes, unless their sum is greater than 180°.

11. Longitude is reckoned east and west of the first meridian from 0° to 180°.

12. *Departure* is the distance east or west that a ship has made from the meridian of the place left, when that distance is so small that the curvature of the earth's surface may be neglected. If the two places are at a great distance from each other, it must be divided into small portions, and the departure is the sum of the departures corresponding to all these portions.



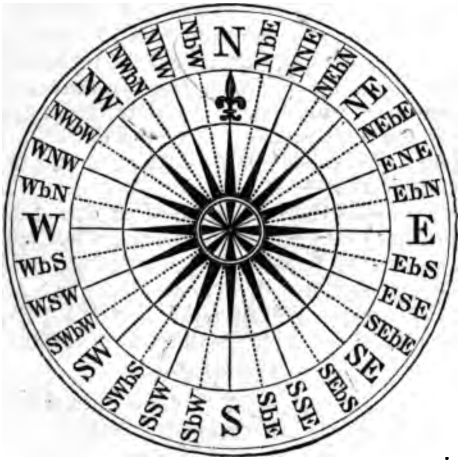
13. If a ship sails due north or south, she keeps on a meridian, makes no departure, and her distance is equal to the difference of latitude.

14. If she sails due east or west, she goes on a parallel of latitude, makes no difference of latitude, and her departure is equal to her distance.

15. The difference of latitude and departure constitute the base and perpendicular of a right-angled triangle, whose hypothenuse is the distance sailed; the angle opposite the departure is the course that the ship makes with the meridian.

A ship is kept on her proper course by means of a

Mariner's Compass.



(26.) This compass consists of a circular card (divided into 360° or 32 points), placed upon a bar of steel, called a needle, to which the magnetic virtue has been imparted. Directly over the needle a straight line is drawn on the card, one end of which is marked N, the other S; and the whole is suspended in a box in such a manner as to maintain a horizontal position, notwithstanding the motion of the ship. The card, supported on the needle by a pivot, moves freely in every direction.

The naming of the points in succession, beginning either at the north or south, and going round the compass either east or west, is called by mariners *boxing the compass.*

The ship's rate of sailing is measured by a *log line,* which is a cord wound round a reel, and divided by knots into equal parts, each of which is the 120th part of a nautical mile. The *log* is a piece of wood in the form of a sector of a circle, the rim of which is loaded with lead, so that when thrown into the sea it assumes a vertical position.

When the log is thrown a sand-glass is turned up, through which the sand passes in thirty seconds, or the 120th part of an hour. When that interval of time has passed, the number of *knots* run off is counted, which is called so many *knots* or miles an hour.

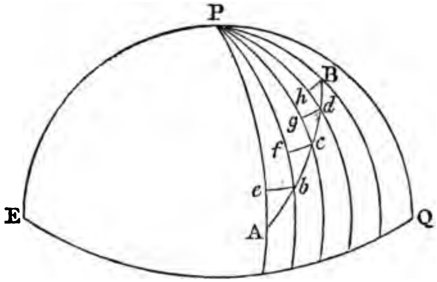
PLANE SAILING.

(27.) *Plane sailing* is the art of navigating a ship upon principles based upon the supposition that the earth is an extended plane.

It has for its objects, primarily, the determining of the *difference of latitude and the departure* which a ship has made between two intervals of time, while running on a certain course at a rate measured by the log. The course, distance, difference of latitude and departure form a right-angled triangle; hence, if any two of these are given, the other two can readily be found.

PROBLEM.

Given the course and distance sailed, to determine the difference of latitude and departure.



When the distance is great, divide AB into small portions, A*b*, *bc*, *cd*, etc. Through the points of division draw the meridians P*b*, P*c*, P*d*, etc.; also draw the parallels of latitude *be*, *cf*, *dg*, B*h*. Then will the angles

eAb, $f b c$, $g c d$, $h d$B, be equal to each other, being the course of the ship.

The distances Ae, bf, cg, dh, are the differences of latitude of A and b, b and c, c and d, d and B.

Also be, cf, dg, Bh, are the departures of the same. Hence, as the difference of latitude of A and B is equal to the sum of these partial differences of latitude, and as the departure of A and B is equal to the sum of the partial departures, we shall have

$$\text{diff. of lat.} = \text{A}e + bf + cg + dh +, \text{ etc.}$$
$$\text{departure} = be + cf + dg + \text{B}h +, \text{ etc.}$$

But, considering these small portions as right lines, we have, from Plane Trig., Theor. 1,

$$\text{A}e = \text{A}b \times \text{cos. course, and } be = \text{A}b \times \text{sin. course.}$$
$$bf = bc \times \text{cos. course, “ } cf = bc \times \text{sin. course.}$$
$$cg = cd \times \text{cos. course, “ } dg = cd \times \text{sin. course.}$$
$$dh = d\text{B} \times \text{cos. course, “ B}h = d\text{B} \times \text{sin. course.}$$

Adding these equations together, member by member, we obtain

$$\text{A}e + bf + cg + dh = (\text{A}b + bc + cd + d\text{B}) \text{ cos. course.}$$
$$be + cf + dg + \text{B}h = (\text{A}b + bc + cd + d\text{B}) \text{ sin. course.}$$

But

$$\text{A}e + bf + cg + dh = \text{difference of latitude,}$$
$$be + cf + dg + \text{B}h = \text{departure,}$$

and $\text{A}b + bc + cd + d\text{B} = \text{distance sailed.}$

Therefore we have

$$\text{diff. of latitude} = \text{distance} \times \text{cos. course.}$$
$$\text{departure} = \text{distance} \times \text{sin. course.}$$

EXAMPLES.

A ship, from latitude 40° 42′ north, sails S.W. by W. 488 miles. Required the latitude she is in and the departure she makes.

Solution:

log. diff. of lat. = log. dist. + log. cos. course − 10.

log. departure = log. dist. + log. sin. course − 10.

distance 488 log.	2.688420		2.688420
course 5 pts. = 56° 15′ cos.	9.744739	sin.	9.919846
	2.433159		2.608266

Diff. of latitude = 271.1. Departure = 405.8.
= 4° 31′.

As the ship is in north latitude sailing southward, we must subtract the difference of latitude from the latitude left, and we shall have the latitude she is in 40° 42′−4° 31′=36° 11′ N., and the departure is 405.8 miles.

2. A ship sails from latitude 21° 13′ north on a N.N.E. course 1000 miles. Required the latitude in and the departure it makes. *Ans.* Diff. of latitude, 15° 23′.9.
Lat. arrived at, 36° 36′.9 N.
Departure, 382.7 miles.

PROBLEM.

(28.) *Given the course and difference of latitude, to find the distance run and departure from the meridian.*

$$\text{distance}=\frac{\text{diff. of lat.}}{\text{cos. course}}=\text{diff. of lat.}\times\text{sec. course.}$$

$$\text{departure}=\text{diff. of latitude}\times\text{tan. course.}$$

EXAMPLES.

1. If a ship sails from latitude 50° 17′ N. on a S.S.W. course, and then, by observation, is in 45° 5′ N., what is the distance run and departure?

log. dist.=log. diff. of lat.+log. sec. course−10.
log. dep.=log. diff. of lat.+log. tan. course−10.

diff. of lat.=5° 12′=312 miles, log. = 2.494155
course=22° 30′ sec. =10.034385
distance=337.7 miles log. 2.528540

diff. of lat.=312 miles log. =2.494155
course=22° 30′ tan. =9.617224
departure=129.23 log. 2.111379

Hence the vessel has run a distance of 337.7 miles, and her departure from the meridian of the former place is 129.23 miles westerly.

2. If a ship sails from a point on the equator N.W. by N., and then, by observation, is in 4° 31′ N. latitude, required the distance run and departure.

Ans. Distance, 487.8 miles.
Departure, 405.6 miles westerly.

PROBLEM.

(29.) *Given the course and departure, to find the distance and difference of latitude.*

$$\text{distance}=\frac{\text{departure}}{\text{sin. course}}=\text{departure}\times\text{cosec. course.}$$

$$\text{diff. of lat.}=\text{departure}\times\text{cot. course.}$$

EXAMPLES.

1. If a ship sails N.E. by E. ¾ E. from a port in 3° 15′ S. latitude until her departure from her first meridian is 406 miles, what are the distance sailed and latitude she is in?

$$\text{log. dist.}=\text{log. dep.}+\text{log. cosec. course.}$$

$$\text{log. diff. of lat.}=\text{log. dep.}+\text{log. cot. course.}$$

Ans. Distance, 449.1 miles, and the
Lat. she is in, 0° 03′ S.

2. If a ship sails N. by W. from a port in 15° 55′ S. latitude until her departure from her first meridian is 225 miles, I demand the distance sailed, and the latitude she is in. *Ans.* Distance sailed, 1153.28 miles.
Lat. she is in, 2° 56′ N.

PROBLEM.

(30.) *Given the distance and difference of latitude, to find the course and departure.*

$$\text{cos. course}=\frac{\text{diff. of latitude}}{\text{distance}}.$$

$$\frac{BC}{\text{sin. }A}=AC.\quad\therefore\ \text{departure}=\text{distance}\times\text{sin. course.}$$

EXAMPLES.

1. A ship sails from a port in 2° 52′ S. latitude in a course southeasterly 488 miles, and then, by observation, is in 7° 23′ S. latitude; what course has she steered and what departure has she made?

Solution. From the latitude by observation take the latitude left; the remainder is the difference of latitude. Then, log. cos. course = log. diff. lat. + ar. co. log. dist.
log. departure = log. dist. + log. sin. course.

diff. of lat. = 4° 31′ = 271 miles log. 2.432969
distance = 488 " ar. co. 7.311580
course 56° 15′ cos. 9.744549
log. distance 488 miles 2.688420
course 56° 15′ sin. 9.919846
405.8 miles. 2.608266

Hence the course is S.E. by E., and the departure 405.8 miles.

2. Suppose a ship sails 1000 miles between the north and east, from a port in 21° 13′ N. latitude, and then, by observation, is in 36° 37′ N. latitude, what course has she steered and what departure has she made?

Ans. Course is N.N.E.
Departure, 382.7 miles.

PROBLEM.

(31.) *Given the distance and departure, to find the course and difference of latitude.*

$$\text{sin. course} = \frac{\text{departure}}{\text{distance}}.$$

$$AB = AC \text{ cos. } A.$$

$$\therefore \text{diff. of latitude} = \text{distance} \times \text{cos. course.}$$

EXAMPLES.

1. A ship sails 488 miles between the north and west, from the latitude of 32° 25′ N., until her departure is 406 miles. What course has she steered and what latitude is she in?

Solution:

log. sin. course = log. dep. + ar. co. log. distance.
log. diff. of lat. = log. dist. + log. cos. course − 10.

dep. = 406 log. 2.608526 dist. log. 2.688420
dist. = 488 ar. co. 7.311580 cos. 56° 18′ 9.744171
56° 18′ sin. 9.920116 2.432591
diff. of latitude = 270.7 miles.

Hence the course is N. 56° 18′ W., or N.W. by W. nearly.

To the latitude left, 32° 25′, add the difference of latitude, 4° 30′.7, and we have 36° 55′.7, the latitude the ship is in.

2. Suppose a ship sails 478.8 miles southeasterly from a port in 1° 45′ N. latitude until her departure is 405.6 miles, what course has she steered and what latitude is she in? *Ans.* The course is S.E. by E.
Latitude in, 2° 46′ S.

PROBLEM.

(32.) *Given the difference of latitude and departure, to find the course and distance.*

$$\frac{BC}{AB}=\text{tan. A.}\quad \therefore \text{tan. course}=\frac{\text{departure}}{\text{diff. of lat.}}.$$

$$\frac{AB}{AC}=\text{cos. A.}\quad \therefore \text{dist.}=\text{diff. of lat.}\times\text{sec. course.}$$

1. A ship sails northeasterly until her difference of latitude is 192 miles, and her departure is 406 miles. What are her course and distance?

log. tan. course=log. dep.+ar. co. log. diff. of lat.

log. distance=log. diff. of lat.+log. sec. course−10.

departure =406 log.	2.608526
diff. of latitude=192 " ar. co.	7.716699
64° 41′ 25″ tan.	10.325225
diff. of latitude=192 log.	2.283301
course 64° 41′ 25″ sec.	10.369052
distance=449.1	2.652353

Hence her course is N.E. by E. ¾ E.; distance, 449.1 miles.

2. A ship sails northwesterly till her difference of latitude is 271 miles, and her departure is 406 miles. Required her course and distance.
Ans. Course, N. 56° 17′, or N.W. by W.
Distance, 488.2 miles.

EXAMPLES ON THE FOREGOING ARTICLES.

1. A ship, in latitude 43° 38′ N., sails N. by E. 267 miles. What latitude has she arrived at, and what is her departure? *Ans.* Latitude in, 48° N.
Departure, 52.08 miles.

2. A ship sails S.S.W. from a port in 41° 30′ N. latitude, and then, by observation, is in 36° 57′ N. latitude. What are the distance run and departure made?
Ans. Distance =295.49 miles.
Departure=113.09 "

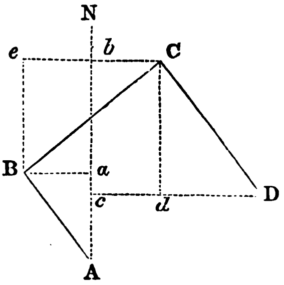
3. A ship, in latitude 3° 52′ S., is bound to a port bearing N.W. by W. $\frac{1}{2}$ W. in latitude 4° 30′ N. How far does the port lie to the westward, and what is the distance from it? *Ans.* Departure = 932.2 miles.
Distance = 1065 "

4. A ship sails S.W. by S. from the latitude of 48° 17′ N., when by observation she is in 46° 17′ N. What distance has she sailed, and what is her departure from her first meridian? *Ans.* Distance = 144.3 miles.
Departure = 80.18 "

5. A ship sails from 3° 25′ S. in a direct course N.W. by N. a distance of 768 miles. Required the latitude and departure. *Ans.* Latitude in, 7° 14′ N.
Departure, 426.67 miles.

TRAVERSE SAILING.

(33.) A traverse is an irregular track which a ship makes by sailing on different courses. These courses, when the distances are so small that the curvature of the earth may be neglected, can be reduced to a single course by means of formulas.



Suppose a vessel starts from A, and sails first to B, thence to C, thence to D; the difference of latitude of AB will be Aa, the departure aB; the difference of latitude of the points BC will be Be=ab; the departure = eC = eb + bC = Ba + bC; and the difference of latitude of C and D will be Cd = bc, and the departure = dD.

Hence *the difference of latitude* of AD is equal to the difference of the sum of the differences of latitude of the north courses, and the sum of the differences of latitude of the south courses. Also,

The *departure* is equal to the difference between the sum of the eastern departures and the sum of the western departures.

Make a traverse table consisting of six columns.

Write over the 1st column, at the left side, Courses; 2d, Distance; 3d, N.; 4th, S.; 5th, E.; 6th, W.

Place the courses and distances run in their proper columns. Find the difference of latitude and departure for each of these courses, and place them in their proper columns. Add up the columns of *northings*, *southings*, *eastings*, and *westings*. Take the difference between the N. and S. columns; also the difference between the columns E. and W. The former difference will be the difference of latitude, the latter will be the departure.

With this difference of latitude and departure, the course and distance made good can be found.

EXAMPLES.

1. A ship sails on several different courses, viz.: 1st, S.W. by W., 27 miles; 2d, W.S.W. ½ W., 30 miles; 3d, W. by S., 25 miles; 4th, W. by N., 18 miles; 5th, S.S.E., 71 miles; 6th, S.S.E. ¾ E., 27 miles; 7th, S. by E., 25 miles; 8th, S., 31 miles. Find the course and distance of the place at which she arrives from that from which she started.

No.	Courses.	Distances.	N.	S.	E.	W.
1.	S.W. by W.	27	...	15.0	......	22.4
2.	W.S.W. ½ W.	30	...	8.7	......	28.7
3.	W. by S.	25	...	4.9	......	24.5
4.	W. by N.	18	3.5	......	......	17.7
5.	S.S.E.	71	...	65.6	27.1	......
6.	S.S.E. ¾ E.	27	...	23.2	13.9	......
7.	S. by E.	25	...	24.5	4.9	......
8.	South.	31	...	31.0	......	......
			3.5	172.9 3.5	45.9	93.3 45.9
			Diff. of lat.	...169.4	Departure	...47.4

Calculation of the above table:

dist. 27 miles log.	1.43136
course S.W. by W.=56° 15′ cos.	9.74474
diff. of lat.=15.0.	1.17610
dist. 27 miles log.	1.43136
course S.W. by W.=56° 15′ sin.	9.91985
dep. 22.4.	1.35121

In like manner, the difference of latitude and departure for each course are calculated; or they may be taken from the Traverse Table, as in Prob. 1, Surveying.

To calculate the course and distance:

departure=47.4 log.	1.67578
diff. of latitude=169.4 ar. co.	7.77109
S. 15° 38′ W. tan.	9.44687
diff. of latitude=169.4 log.	2.22891
course 15° 38′ sec.	10.01637
distance=175.9.	2.24528

Hence the course is S. 15° 38′ W.; distance, 175.9 miles.

2. A ship sails on several different courses, viz.: 1st, S.S.E., 15 miles; 2d, S.E., 34 miles; 3d, W. by S., 16 miles; 4th, W.N.W., 39 miles; 5th, S. by E., 40 miles. Required the course and distance made good.

Ans. Course, S. 12° 13′ W.
Distance, 66.8 miles.

PARALLEL SAILING.

(34.) When a ship sails exactly east or exactly west, she continues on a parallel of latitude. This is called *parallel sailing.* The ship remains in the same latitude, and her distance is equal to her departure.

The object of the present chapter is to determine the difference of longitude made by a vessel thus sailing, and the distance between two places situated on the same parallel of latitude whose longitudes are given.

Imagine the earth to be cut by two planes perpendicular to the axis on which it turns, one passing through the equator, the other through a parallel of latitude. Hence, by Geometry, as the radius of the one is to the radius of the other section, so is the circumference of the one to the circumference of the other; or, as any arc of the circumference of the one to the same arc of the circumference of the other. But the arc of the circumference of the equator is the longitude, and the arc of a parallel is the departure.

Taking the radius at the equator equal to unity, the radius of the plane passing through a given latitude will be the cosine of that latitude. Therefore,

as $1 : \text{cos. lat.} :: \text{diff. of long.} : \text{departure}$;
or, $\text{diff. of long.} \times \text{cos. lat.} = \text{departure}.$

$$\therefore \text{diff. of long.} = \frac{\text{departure}}{\text{cos. lat.}} = \text{departure} \times \text{sec. lat.}$$

Or, *as cos. lat. of the parallel is to radius, so is the distance sailed to the difference of longitude.*

EXAMPLES.

1. Suppose a ship starts from Sandy Hook, in latitude 40° 28′, longitude 74° 8′ W., and sails directly east 136.4 miles; required the difference of longitude, and longitude in.

departure = distance = 136.4 log. 2.13481
latitude = 40° 28′ secant 10.11874
diff. of long. = 179.3. 2.25355

longitude left = 74° 8′ W.
diff. of longitude = 2° 59′
longitude in = 71° 9′

2. Required the distance of St. Anthony's Cape (Newfoundland) from Cape Clear (Ireland):

lat. of St. Anthony's Cape, 51° 20′ N., long. 55° 36′ W.
lat. of Cape Clear . . . 51° 20′ N., long. 9° 37′ W.
diff. of long. . . . . 45° 59′ W.

diff. of long. = 45° 59′ = 2759 miles log. 3.440752
lat. = 51° 20′ cos. 9.795733
dist. = dep. = 1723.3 miles. 3.236485

3. A ship, in latitude 53° 36′ N., longitude 10° 18′ W., sails due west 236 miles. Required the difference of longitude, and longitude in.

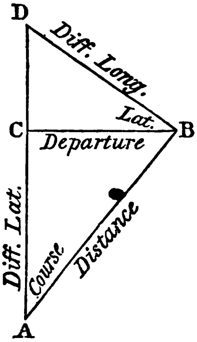
Ans. Difference of longitude, 6° 38′.
Longitude in . . . 16° 56′ W.

4. Required the distance of Seville (Spain) from Cape Henry (Virginia), latitude = 36° 59′ N.: Longitude of Seville = 5° 58′ W.; longitude of Cape Henry = 76° 21′ W. *Ans.* 3373 miles.

MIDDLE-LATITUDE SAILING.

(35.) The principle on which the calculations of middle-latitude sailing are founded is this: The departure is calculated, and the longitude deduced therefrom by the Rules of *Parallel Sailing*, using for the latitude the arithmetical mean between the latitude sailed from and the latitude arrived at.

The difference of longitude found by this method is less than the true one, thus tending to make the ship's calculated place behind that in which she really is, certainly a *less safe* error than the opposite one; and when either the middle latitude or the difference of latitude is large, the error is altogether too great to render it safe to employ this method unless a proper correction be applied to it. Middle-latitude sailing is, however, made accurate by applying to the middle latitude a *correction* which is given in Loomis's Tables, page 149.



The middle latitude is equal to half the sum of the two latitudes when they are both of the same name, and to half their difference when one is north and the other south.

By combining the triangle ABC (which represents the course, distance, difference of latitude, and departure) with the triangle BCD (parallel sailing), we obtain a triangle ABD, by which all the cases of middle-latitude sailing can be solved.

1°. In the triangle ABD we have

sin. D : sin. A :: AB : BD; that is,

as the *cosine of the middle latitude is to the sine of the course, so is the distance sailed to the difference of longitude.*

2°. In the triangle BCD we have

cos. CBD : radius :: BC : BD; that is,

as the *cosine of the middle latitude is to radius, so is the departure to the difference of longitude.*

3°. In the triangle ABC we have

AC : BC :: radius : tan. A.

Combining this with the second proportion, we shall have cos. CBD : tan. A :: AC : BD; that is, as the *cosine of the middle latitude is to the tangent of the course, so is the difference of latitude to the difference of longitude.*

EXAMPLES.

1. Required the distance between Cape Cod Lighthouse, latitude 42° 5′ N., longitude 70° 4′ W., and the island of St. Mary, latitude 36° 59′ N., and longitude 25° 10′ W.:

the difference of latitude is	5° 6′=306′
the difference of longitude is	44° 54′=2694′
the middle latitude is	39° 32′
to which add the correction, p. 149	3′
the corrected middle latitude is	39° 35′

Hence, to find the course, we have

as	the diff. of latitude 306′ ar. co.	7.514279
	: the diff. of longitude 2694′ log.	3.430398
	:: cos. middle latitude 39° 35′	9.886885
	: tan. course 81° 37′.	10.831562

To find the distance:

As	sin. of the course 81° 37′ ar. co.	0.004666
	: cos. middle latitude 39° 35′ log.	9.886885
	:: diff. of longitude 2694′	3.430398
	: distance 2098.7.	3.321949

2. A ship, in latitude 49° 57′ N., and longitude 30° W., sails S. 39° W., and, by observation, the latitude is found to be 45° 31′ N.; required the distance run and longitude she is in.

As cos. course : radius :: diff. lat. : distance=342.27.

cos. middle lat.* : tan. course :: diff. lat. : diff. long.= 320′.4=5° 20′. ∴ long. in=35° 20′ W.

3. A ship sails from latitude 42° 30′ N., and longitude 58° 51′ W., S.W. by S. 33° 45′, 591 miles. Find her latitude, and the longitude she is in.

Ans. Latitude in, 34° 19′ N.
Longitude in, 65° 51′ W.

---

* The middle latitude must always be corrected.

4. A ship sails E.N.E. from latitude 15° 37′ N. to the latitude 17° 53′ N. Required her difference of longitude. *Ans.* 5° 43′.

MERCATOR'S SAILING.

(36.) By the theory of Parallel Sailing we have

cos. latitude : R :: dist. : diff. of longitude.

But cos. : R :: R : sec.

∴ 1′ of a parallel : 1′ of a meridian :: R : sec. lat.

Hence, if the meridians are supposed to be parallel to each other, as they are on Mercator's Chart, the degree or minute of latitude must be increased in proportion to the secant of the latitude.

Therefore, if the radius be supposed to be equal to one mile, the length of the first mile of latitude from the equator will be represented by the secant of 1′; the second mile, by the secant of 2′; the third mile, by the secant of 3′, etc.

The table of meridional parts may be found by adding together the minutes thus obtained.

Mer. parts of 1′=sec. 1′.

Mer. parts of 2′=sec. 1′+sec. 2′, etc., etc.

And from this table *Mercator's Chart* may be constructed. Thus:

Draw a horizontal line to represent the equator or any parallel of latitude; then from a scale of equal parts take a distance to represent the number of degrees through which the meridians are to pass; lay this off as many times as are necessary, and through these points draw a series of parallels to form the meridians, all perpendicular to the horizontal line.

Then, for the next parallel of latitude, find from the table of meridional parts the meridional difference of latitude corresponding to the degrees between the first and second parallel, and lay off this distance for the interval between the two parallels.

Then find the meridional difference of latitude between this and the next, and lay it off in the same manner, and so on. Then through these points draw lines parallel to the horizontal line to represent parallels of latitude.

Take from a table of latitudes and longitudes of places the latitude and longitude of each particular place contained within the bounds of the chart, and lay a ruler over its latitude, and another crossing that over its longitude, the points where these lines meet will be the place upon the chart. The most remarkable points of the sea-coast being thus laid down, lines may be drawn from point to point. This chart is not to be considered a true representation of the earth's surface, for the figures of islands and countries are distorted toward the poles, as is evident from the construction; but the degrees of latitude and longitude are increased in the same proportion, so that the bearings between places will be the same on the chart as on the globe; and, as the meridians are right lines, it follows that the rhumbs which form equal angles with the meridians will be straight lines, which renders this projection much more easy for the mariner than any other.

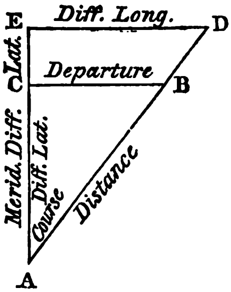
(37.) The principles of *Mercator's Sailing* are strictly true; but there are errors in its use, which are caused by the defects of the Tables of *Meridional Parts.*

These tables are computed to the nearest mile only, and the consequence is that the difference between two meridional parts will sometimes have an error of nearly a whole mile. Now the formula in common use,

R × *diff. long.* = *mer. diff. lat.* × *tan. course,*

shows that when the course is greater than 45°, whatever error there may be in the meridional difference of latitude, it will be increased in finding the difference of longitude; and when the course approaches to 90°, the error is multiplied to an alarming extent.

If with the course, distance, difference of latitude, and departure we form the triangle ABC, then produce AC to E, making AE equal to the meridional difference of latitude, DE will represent the difference of longitude corresponding to the departure CB.



Now, from the similar triangles ABC, ADE, we have

AC : AE :: CB : ED;

that is, *the true difference of latitude is to the meridional difference of latitude as the departure is to the difference of longitude.*

Again, in the triangle ADE we have

R : tan. A :: AE : ED;

or, *as radius is to the tangent of the course, so is the meridional difference of latitude to the difference of longitude.*

Or the formula

R. diff. long.=mer. diff. lat.×tan. course.

EXAMPLES.

1. A ship, from latitude 49° 57′ N., and longitude 30° W., sails S. 39° W. till she arrives in the latitude of 45° 31′ N. Required the distance run and longitude in.

latitude left	49° 57′ N.	mer. parts	3470
latitude in	45° 31′ N.	mer. parts	3074
Diff of lat.=	4° 26′=266′.	mer. diff. lat.=	396

As	cos. course 39° ar. co.	0.109497
	: prop. diff. of latitude 266′	2.424882
	:: radius	10.000000
	: distance 342.3.	2.534379
As	radius	10.000000
	: mer. diff. of latitude 396′	2.597695
	:: tan. course 39°	9.908369
	: diff. of longitude 320.7.	2.506064

longitude left	30° 00′ W.
diff. of long. 320.7=	5° 21′ W.
longitude in	35° 21′

2. What are the bearing and distance from Sandy Hook, latitude 40° 28′ N., longitude 74° 1′ W., to Havre, latitude 49° 29′ N., longitude 0° 6′ E.?

Ans. N. 80° 13′ E.; distance, 3183.8 miles.

3. Find the course and distance from the east point of St. Michael's, latitude 37° 48′ N., longitude 3° 38′ W., to the Start Point, latitude 50° 13′ N., longitude 25° 13′ W. *Ans.* N. 51° 11′ 14″ E.; distance, 1189 miles.

MISCELLANEOUS EXAMPLES.

1. A ship sails northwesterly from Callao, latitude 12° 2′ S., long. 77° 4′ W., until she arrives at the latitude 23° 7′ N., and has made a departure of 9983 miles. At what place has she arrived? *Ans.* Canton.

2. A ship sails from latitude 49° 57′ N., longitude 30° W., on a course S. 39° W., and is then in the latitude 45° 31′ N. Find the distance run and longitude in.

Ans. Distance, 342.3 miles.
Longitude in, 35° 20′ W.

3. A ship near the Cape of Good Hope, in latitude 34° 26′ S., longitude 18° 23′ E., sails N. 46° 59½′ W., 6747 miles. At what place has she arrived? *Ans.* Boston.

4. What are the bearing and distance of San Francisco, latitude 37° 49′ N., longitude 122° 14′ W., from New York, latitude 40° 42′ N., and longitude 73° 59′ W.?

Ans. Bearing, S. 85° 35′ W.
Distance, 2246.5 miles.

5. A ship sails from a point on the equator, in 16° W. longitude, a direct course S.S.W. a distance of 3635 miles. What are the latitude and longitude in?

Ans. Latitude, 55° 58′ S.
Longitude, 44° 06′ W.

6. A ship sails from latitude 41° N., and longitude 52° W., southeasterly a distance of 2896 miles, making a departure of 875 miles. What is her position?

Ans. 13 miles east of Cape St. Roque.

The following Rule for determining the longitude, instead of using that of middle latitude or Mercator's sailing, is inserted here more for the benefit of practical navigators than for students. The principles involved in obtaining the formulas are of a higher order of mathematics than the students have yet attained at this stage of their progress. The method of obtaining the formulas is given in a note below.

RULE.

Take the difference between the logarithmic tangents of half the complements of the latitude left and the latitude arrived at.

To the logarithm of this difference add the logarithmic tangent of the course and the constant logarithm 3.898489; *reject ten, and the remainder will be the logarithm of the difference of longitude in nautical miles.*

The two complements must be taken from the same pole; that is, if the latitudes are both north or both south, their complements will be found by either *subtracting each from* 90° *or adding each to* 90°; but if one be north and the other south, *one must be taken from* 90°, and *the other added to* 90°.

EXAMPLES.

1. A ship sails from St. Augustine, latitude 29° 52′ N., longitude 81° 25′ W., upon a course N.E. by E. until her latitude is found to be 34° 40′ N. Required the longitude in and distance sailed.

Here both latitudes are north; the complements are therefore found by subtracting each from 90°.

$$\begin{array}{r} 90^\circ \\ 29^\circ\ 52' \\ \hline 2)\,60^\circ\ 08' \\ \hline 30^\circ\ 04' \end{array} \qquad\qquad \begin{array}{r} 90^\circ \\ 34^\circ\ 40' \\ \hline 2)\,55^\circ\ 20' \\ \hline 27^\circ\ 40' \end{array}$$

30° 04′ log. tan.	9.762606
27° 40′ log. tan.	9.719555
	$\delta =$ .043051
$\delta =$ .043051 log.	$\bar{2}.633974$
constant log.	3.898489
course = 56° 15′ tan.	10.175107
diff. of long. 510′ = 8° 30′.	2.707570

longitude left	81° 25′
diff. of longitude	8° 30′
	72° 55′ W.

To find the distance:

diff. of latitude 288 log.	2.459392
course 56° 15′ sec.*	10.255260
distance=518.4.	2.714652

2. A ship sailed from Cape Horn, in latitude 55° 58′ S., longitude 67° 21′ W., on a direct course N. 28° 22′ E., and, by observation, found the latitude to be 49° 29′ N. Required the difference of longitude and the place arrived at.

Here one latitude is *south*, the other *north;* we therefore add the south latitude to 90°, and subtract the north from 90°, for the complements.

90°	90°
55° 58′	49° 29′
2)145° 58′	2)40° 31′
72° 59′	20° 15½′

72° 59′ log. tan.	10.514209
20° 15½′ log. tan.	9.567126
	$\delta =$ .947083

$\delta =$ .947083 log.	$\bar{1}.976390$
constant log.	3.898489
course=28° 22′ tan.	9.732351
diff. of long.=4047 log.	3.607230

diff. of longitude	=67° 27′ E.
longitude left	=67° 21′ W.
longitude arrived at	= 0° 6′ E.

Hence the ship has arrived at Havre de Grace.

3. Determine the bearing and distance from Sandy Hook, latitude 40° 28′ N., longitude 74° 1′ W., to the Cape of Good Hope, latitude 34° 22′ S., longitude 18° 30′ E.

90°	90°
40° 28′	34° 22′
2)49° 32′	2)124° 22′
24° 46′	62° 11′

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[*] The logarithmic *secant* is found by subtracting the logarithmic *cosine* from 20.

62° 11′ log. tan.	10.277685
24° 46′ log. tan.	9.664039
	$\delta =$ .613646
$\delta =$.613646 log.	$\bar{1}$.787917
constant log.	3.898489
diff. of long. 5551 ar. co. log.	6.255629
S. 48° 48′ 45″ cot.	9.942035

To find the distance:

diff. of latitude 4490 log.	3.652246
course 48° 48′ 45″ sec.	10.181428
distance=6808.26 log.	3.833674

4. A ship, in latitude 42° 30′ N., and longitude 58° 51′ W., sails S.W. by S., and finds, by observation, the latitude she is in to be 38° 21′ N. Required the longitude arrived at. *Ans.* 62° 29½′ W.

5. Determine the course and distance on a rhumb line extending along the level surface of the earth from Washington City, latitude 38° 58′ N., longitude 77° 2′ W., to Boston, in latitude 42° 22′ N., longitude 71° 4′ W.

Ans. Course, N. 53° 04′ 28″ E.
Distance, 339.55 miles.

If the course of the ship could be accurately determined by the compass, the above method would invariably show its position on the ocean. But this is not always the case. There are imperfections in steering, variations of the needle, effects of unknown currents, and many other unavoidable sources of error, all of which render the place of the ship more or less doubtful.

Hence, to determine her position with the precision and certainty which the safety of her crew and cargo require, recourse must be had to other methods than that of the *dead reckoning*. The science is then called *Nautical Astronomy*.

Investigation of the above Formulas.

If we take the pole of the earth as the origin of *polar spherical coordinates*, and the first meridian as the angular axis, the radius vector of any point on the surface of the globe will evidently be the complement of the latitude of that point, and the angle which this radius vector makes with the first meridian will be the longitude of that point.

Let r_1=the radius vector, or complement of latitude at the ship at any time,
ϕ_1=the longitude of the same place, or the angle which the radius vector makes with the first meridian.
a=the angle which the ship's course makes with the radius vector.
R=radius of the earth in nautical miles.

Then, from the principles of *Analytical Spherical Geometry*, we shall have

$$d\phi_1=-\text{R tan. } a\,\frac{dr_1}{\text{sin. } r_1},\ \text{or}\ \phi_1=-\text{R tan. } a\int\frac{dr_1}{\text{sin. } r_1}.$$

$$\therefore \phi_1=-\text{R tan. } a \text{ log. tan. } \tfrac{1}{2}r_1+c$$ (Calculus, p. 271, Ex. 106).

If we take r_2=radius vector of another point on the same rhumb line, and ϕ_2=longitude of that point, then

$$\phi_2=-\text{R tan. } a \text{ log. tan. } \tfrac{1}{2}r_2+c.$$

$$\therefore \phi_1-\phi_2=\text{R tan. } a\,(\text{log. tan. } \tfrac{1}{2}r_2-\text{log. tan. } \tfrac{1}{2}r_1).$$

These are Naperian logarithms, and to convert them into common logarithmic tangents we must multiply by the modulus of the system, m, which gives us

$$\phi_1-\phi_2=\text{R}m \text{ tan. } a\,(\text{log. tan. } \tfrac{1}{2}r_2-\text{log. tan. } \tfrac{1}{2}r_1) \quad . \quad . \quad (A)$$

But $\phi_1-\phi_2$=difference of longitude,
tan. a=tangent of the course,
$\tfrac{1}{2}r_2=\tfrac{1}{2}\,(90^\circ-l)$, where l denotes the less latitude,
$\tfrac{1}{2}r_1=\tfrac{1}{2}\,(90^\circ-\text{L})$, where L denotes the greater latitude,
$\text{R}m=\frac{10800'}{3.14159}\times 2.302585=7915.704467$, and the
log. of Rm=3.898489.

If we now put δ=log. tan. $\tfrac{1}{2}\,(90^\circ-l)$−log. tan. $\tfrac{1}{2}\,(90^\circ-\text{L})$, Equation A becomes

diff. of long.=Rm × tan. course × δ . . . . (B)
or log. diff. long.=3.898489+log. tan. course+log. δ−10 . (C)
and log. cot. course=3.898489+log. δ+ar. co. log. diff. long. (D)
log. distance=log. diff. lat.+log. sec. course−10 . . (E)

whence the truth of the above rules is manifest.

THE END.